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On the global instability of free disturbances with a time-dependent nonlinear viscous critical layer

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A theoretical study is made of the global nonlinear growth or decay, in space and time, of an unsteady non-neutral disturbance/wavepacket when a time-dependent nonlinear viscous critical layer is present. The basic flow considered is a steady quasiparallel channel flow, boundary layer or liquid-layer flow at high Reynolds number. The unsteadiness with regard to the critical layer shows itself less in the internal dynamics than in the relatively slow movement of the layer across the flow, the temporal and spatial rate of movement discussed being sufficient to affect the nonlinear viscous balance in the layer. This greatly reduces the mean-flow distortions produced. The disturbance amplitude, in contrast, responds nonlinearly on faster time- and space-scales, both inside and outside the critical layer. These slower- and faster-scale properties inside the critical layer and outside, i.e. globally, are coupled together in general. The work addresses first the structure and nonlinear evolution equations for the growing or decaying free disturbance and the critical layer. But preliminary analysis in special cases suggests, among other things, the significant result that previous nonlinear studies based on quasineutral assumptions give unstable subcritical threshold amplitudes, above which increasingly fast disturbance growth takes place globally.

1. Introduction

Our concern here is with the nonlinear growth or decay, in time and space, of an initially small disturbance to a given parallel or almost parallel flow at high Reynolds number. At such Reynolds numbers the critical layer, where the basic flow speed coincides with the local effective wavespeed, can play a vital role in determining the behaviour of the disturbance, particularly when nonlinearity exerts a significant influence. The effects of increased nonlinearity as the disturbance size increases are felt primarily within the critical layer itself, which gradually undergoes a change in character. There have been many theoretical studies made of the properties of nonlinear critical layers (e.g. Benney & Bergeron 1969; Davis 1969; Haberman 1972, 1976; Brown & Stewartson 1978, 1980; Smith & Bodonyi 1982*a*,*b*; Bodonyi, Smith & Gajjar 1983), while Stewartson (1981) has given an interesting review of many aspects of the subject. To the best of our knowledge, however, most theoretical efforts so far (apart from purely inviscid theory, as noted later) have been focused on disturbances that are neutral travelling waves outside of the critical layer, with non-neutral states of growth or decay having been considered only with respect to the motion within the critical layer. This amounts to an assumption or restriction that the critical layer is *fixed*.

In this paper the aim is to remove that restriction, for a fluid of small but non-zero viscosity, in order to discuss theoretically the temporal–spatial growth or decay of the disturbance amplitude globally *throughout the whole flowfield*, not solely inside the critical layer. It would seem that such a discussion is almost certainly more relevant to the behaviour of the unsteady disturbances observed experimentally in boundary layers, channel flows and liquid layers for instance. The study is also of interest as a test on the tentative suggestions of Smith & Bodonyi (1982*a*) and Bodonyi *et al.* (1983), which are based on a quasineutral assumption of only very gradual growth indeed. They suggest formally that a very slowly increasing disturbance amplitude can lead to amplitude-dependent instabilities occurring with shortened wavelengths, enhanced wavespeeds and (hence) movement of the critical layer, and that, in line with Benney & Bergeron's (1969) suggestions, this may be related ultimately to the shedding of vorticity across and out of a disturbed boundary layer. A similar approach for three-dimensional disturbances suggests (Smith & Bodonyi 1982*b*) amplitude-dependent instabilities arising in Hagen–Poiseuille flow. So the evolution of a nonlinear disturbance or wavepacket, rather than the succession of nonlinear neutral states considered previously, is our concern. Associated with the space–time evolution of the disturbance through the flowfield, the effects of a *moving* critical layer have to be taken into account, and these prove to be of some consequence, more so in some ways than unsteady effects confined within the critical layer. For, as the critical layer is relatively thin, its actual movement across the flow can produce more change in the internal flow properties than does the inherent unsteadiness of the disturbance velocity. In addition, a relatively slow movement of the critical layer position is found (below) to be accompanied by a much faster temporal and spatial response in the typical disturbance amplitude, inside and outside the critical layer. The study below concentrates on the (upper-branch) structure, governing equations and certain preliminary properties of both the non-stationary critical layer and the time-dependent amplitude.

Section 2 discusses the basic principles involved when a disturbance is 'unsteady', i.e. non-neutral, growing or decaying in time and space, in the presence of a nonlinear critical layer. The importance of even slight critical-layer movement is confirmed, and leads to a rather general equation describing the behaviour of the time-dependent nonlinear critical layer: although, to be sure, certain important elements of the unsteady theory here can be found in the interesting works by Dickinson (1970), Benney & Maslowe (1975), Béland (1978), Stewartson (1981), Cowley (1981) and Benney (1983) and others, largely for inviscid fluids. The multiple scales introduced in §2 are supplemented in §3, where the specific contexts of the attached boundary layer, channel flow and liquid-layer flow are considered in turn, and the faster-scale response of the amplitude becomes evident. As a result a set of coupled nonlinear evolutionary equations, controlling the amplitude and effective wavespeed or group velocity of a wave packet, is derived for the motion inside and outside the critical layer. Section 4 describes the solution in the special case of fixed-frequency disturbances. These exhibit a subcritical threshold phenomenon whereby initial disturbances of amplitude above/below the nonlinear threshold value amplify/decay nonlinearly at later times, further downstream. In §5 limiting analytical solutions are considered for the time-dependent nonlinear critical-layer problem itself, which otherwise requires a numerical treatment. One general property is that no significant jump in the mean vorticity is possible across the moving critical layer, so that (as a referee noted) no large mean-flow disturbances are induced outside, not as for fixed critical layers. Finally a brief summary is given in §6. Only two-dimensional disturbances are examined, for an incompressible fluid of constant density ρ_∞ and kinematic viscosity ν_∞. The Reynolds number is written $Re = u_\infty l_\infty \nu_\infty^{-1}$ and is assumed to be asymptotically large. Concerning the practical value of this assumption, see Smith, Papageorgiou & Elliott's (1984) application of the asymptotic theory to finite(even subcritical)-*Re* properties, and the close comparisons with experiments on Tollmien–Schlichting disturbances in Smith (1979); we note in addition that, even though terms with small inverse powers of *Re* may be involved below, the practical value of the asymptotic results depends to a large extent also on the coefficients multiplying those terms, and these coefficients are not known in advance. Here u_∞ and l_∞ are the typical speed and lengthscale of the basic motion, l_∞ being the streamwise development scale of the boundary layer, the thickness of the channel or the width of the liquid layer. The time, Cartesian coordinates, associated velocity components and pressure are $l_\infty u_\infty^{-1} t$, $l_\infty(x, y)$, $u_\infty(u, v)$ and $\rho_\infty u_\infty^2 p$, the stream function being $l_\infty u_\infty \psi$. In the cases considered the basic flow can be taken to be parallel along the x-axis to the order of working necessary.

2. General arguments on unsteady nonlinear critical layers

It seems appropriate to start with the general argument leading to the central time-dependent critical-layer problem below without reference to a specific flow as yet. This delays the use of expansions (necessary eventually) in somewhat strange powers of the Reynolds number, and some of the initial scalings involved can be identified more readily with the classical scalings (e.g. Reid 1965). The formal expansions for an attached boundary layer and channel and liquid-layer flows in particular will be verified in the later sections.

To allow for the slow spatial and temporal variations we introduce the multiple-scales replacement of $\partial/\partial x$ and $\partial/\partial t$ by

$$\left.\begin{aligned} \frac{\partial}{\partial x} &\rightarrow \alpha\frac{\partial}{\partial X} + \Gamma^{(x)}\frac{\partial}{\partial X_2} + \ldots, \\ \frac{\partial}{\partial t} &\rightarrow -\beta\frac{\partial}{\partial X} + \Gamma^{(t)}\frac{\partial}{\partial T_2} + \ldots. \end{aligned}\right\} \tag{2.1}$$

Here α is the wavenumber, $\beta = \alpha c$ is the frequency and c is the wavespeed of the disturbance/non-simple-wave solutions, which are periodic in X but not in the slow variables $x = X_2/\Gamma^{(x)}$ and $t = T_2/\Gamma^{(t)}$, where the scales $\Gamma^{(x)}$ and $\Gamma^{(t)}$ are to be determined.

Consider first, then, the *steady* problem for *infinitesimal* disturbances. The majority of the motion consists of the basic flow $(u, v, p) = (u_B, 0, p_B)$ together with the small disturbances (u_0, v_0, p_0) proportional to e^{iX}. This holds even in the neighbourhood of the Stokes wall layer and the critical layer, which are assumed to be asymptotically distinct but with the critical layer still lying close to the wall. Just outside the critical layer, where the basic flow is comparable to the wavespeed c, the flow expansions take the form

$$\left.\begin{aligned} u &\sim \lambda c_* \bar{Y} + \tfrac{1}{2}c_*^2 \lambda_2 \bar{Y}^2 + \delta(u_0 + c_* u_1 + \ldots), \\ v &\sim \alpha\delta\sigma_* c_*(v_0 + c_* v_1 + \ldots), \\ p &\sim \delta c_*(p_0 + c_* p_1 + \ldots). \end{aligned}\right\} \tag{2.2}$$

Here $c_* \tilde{Y} = \sigma_*^{-1} y$ defines the boundary-layer coordinate and $c = c_0 c_*, \alpha = \alpha_0 \alpha_*$, where the unknown orderings c_*, α_*, σ_* depend on the Reynolds number, whereas c_0, α_0 are $O(1)$. Also δ is the infinitesimal disturbance size and λ is the local skin friction. If the local profile curvature λ_2 is non-zero then a logarithmic singularity in u_2 is induced usually and requires a thin viscous critical layer of thickness $\sigma_*(\alpha_* \sigma_*^2 Re)^{-\frac{1}{3}}$ to smooth out the irregularity. The familiar phase shift of $-\pi$ in the logarithm below the critical layer results. The Stokes layer of thickness of order $\sigma_*(\alpha_* c_* \sigma_*^2 Re)^{-\frac{1}{2}}$ produces a phase shift in the major disturbance and provides the necessary displacement condition to match with the critical-layer shift. This fixes effectively the Reynolds-number dependence of the wavespeed and wavenumber, i.e.

$$c = O[(\alpha_* \sigma_*^2 Re)^{-\frac{1}{5}}]. \tag{2.3}$$

Additional relations dependent on the particular basic flow then determine the neutrally stable modes explicitly.

The above description holds for infinitesimal disturbances. As the disturbance size is increased strongly *nonlinear* interactions first come into play inside the critical layer, and streamwise momentum balances show that the critical-layer properties are altered significantly when δ takes the critical value

$$\delta_c = O\left[\frac{(\alpha_* Re\, \sigma_*^2)^{-\frac{2}{3}}}{c_*}\right]. \tag{2.4}$$

The governing equations in that case are those derived by Haberman (1972). Outside the critical layer the main change produced is that the jump in the streamwise velocity is no longer monochromatic, because higher harmonics are induced, and in particular the phase shift ϕ depends on the disturbance size. Only for $\delta \ll \delta_c$ are classical linear properties recovered, with $\phi \rightarrow -\pi$.

Let us suppose now that *unsteadiness* is also present, in the sense that non-neutral conditions hold and introduce relatively slow time-dependent variations globally in the disturbance amplitude A_0, allowing growing or decaying nonlinear modes to be accommodated. Therefore in the development of the critical layer the wavespeed and wavenumber become slowly varying. The time dependence first becomes important inside the critical layer. Since the critical layer is defined by

$$y = y_c + \sigma_*(\alpha_* \sigma_*^2 Re)^{-\frac{1}{3}}\, Y, \tag{2.5}$$

where $y_c = O(\sigma_* c)$ determines the critical-layer position and is dependent on the slow variables X_2 and T_2, we have in addition to (2.1)

$$\left.\begin{aligned} \Gamma^{(x)} \frac{\partial}{\partial X_2} &\rightarrow \Gamma^{(x)}\left[\frac{\partial}{\partial X_2} - \sigma_*^{-1}(\alpha_* \sigma_*^2 Re)^{\frac{1}{3}} \frac{\partial y_c}{\partial X_2} \frac{\partial}{\partial Y}\right], \\ \Gamma^{(t)} \frac{\partial}{\partial T_2} &\rightarrow \Gamma^{(t)}\left[\frac{\partial}{\partial T_2} - \sigma_*^{-1}(\alpha_* \sigma_*^2 Re)^{\frac{1}{3}} \frac{\partial y_c}{\partial T_2} \frac{\partial}{\partial Y}\right]. \end{aligned}\right\} \tag{2.6}$$

The terms in $\partial/\partial Y$ reflect the *movement* of the critical layer itself and are much larger than the basic slow variations $\partial/\partial X_2$ and $\partial/\partial T_2$ here, simply in view of the thinness of the critical layer. Previous studies on unsteady nonlinear critical layers have concentrated mostly on the case where the $\partial/\partial Y$ contributions in (2.6) are negligible. With a moving critical layer, however, these terms dominate and cannot be neglected. Similar terms appear in Cowley's (1981) mainly inviscid analysis of moving critical layers (as a referee has pointed out; see also below), although in a somewhat different context and in which the basic flow is also time-dependent.

An unsteady–viscous–inertial balance now gives the appropriate scalings when these new effects become significant: namely when

$$\Gamma^{(t)} = O(\alpha_* \delta_c), \quad \Gamma^{(x)} = O\left(\frac{\alpha_* \delta_c}{c_*}\right). \tag{2.7}$$

The inclusion of nonlinearity and time dependence modifies the Haberman equation, and instead the vorticity ζ^* now satisfies the equation

$$\mu\left(1 - \frac{\partial \zeta^*}{\partial Y^*}\right) + Y^* \frac{\partial \zeta^*}{\partial X^*} + \sin X^* \frac{\partial \zeta^*}{\partial Y^*} = \gamma_c \frac{\partial^2 \zeta^*}{\partial Y^{*2}} \tag{2.8}$$

in normalized form. Here, from the inertial forces in the Navier–Stokes equations, with the transformations (2.6),

$$\mu \propto \left(\frac{\partial c_0}{\partial T_2} + c_0 \frac{\partial c_0}{\partial X_2}\right) \Big/ A_0,$$

since the basic flow $u_B \sim c$ and $(\partial/\partial T_2 + c_0\, \partial/\partial X_2)\, y_c = (\partial/\partial T_2 + c_0\, \partial/\partial X_2)\, c_0 (\mathrm{d}y_c/\mathrm{d}c_0)$, to leading order. The case $\mu = 0$ reproduces the steady Haberman equation, and the parameter γ_c is the Haberman parameter gauging viscous against inviscid forces as used in Smith and Bodonyi (1982*a*), except that now it is slowly varying. The extra term $\mu(1 - \partial\zeta^*/\partial Y^*)$ arises because of the slow variations present, which force the critical layer to move. Again, ζ^* satisfies the same boundary conditions as in the steady case, and the leading-order properties outside the critical layer remain largely unaltered.

It should be emphasized that so far the slow temporal and spatial variations (T_2, X_2) appear only parametrically in the time-dependent critical-layer equation (2.8). The properties of this time-dependent equation strongly influence the global amplitude dependence, however, via the velocity jump, although the explicit slow-time and slow-space variations of the amplitude require the examination in §3 below of the higher-order terms, both inside and outside the critical layer. Again, the slow variations in time and space chosen above are those which first significantly alter the steady Haberman problem. It can be expected, though, that the limiting solutions of this time-dependent problem for μ large or small will indicate what the different structures are for alternative spatial and temporal variations, both weaker and stronger. Finally here, we consider whether faster temporal variations but confined within the critical layer can be admitted also, formally, much like those considered by Dickinson (1970), Warn & Warn (1978), Stewartson (1978, 1981) among others. Equation (2.8) is then modified by inclusion of an extra contribution $\partial\zeta^*/\partial T^*$ on the left-hand side, giving the generalized form

$$\frac{\partial \zeta^*}{\partial T^*} + \mu\left(1 - \frac{\partial \zeta^*}{\partial Y^*}\right) + Y^* \frac{\partial \zeta^*}{\partial X^*} + \sin X^* \frac{\partial \zeta^*}{\partial Y^*} = \gamma_c \frac{\partial^2 \zeta^*}{\partial Y^{*2}}, \tag{2.9}$$

where $t = \sigma_*^{\frac{2}{3}} \alpha_*^{-\frac{2}{3}} Re^{\frac{1}{3}} T^*$ defines T^*. There is a dilemma here, however, since for consistency A_0 and the wavespeed must be independent of T^* and indeed, we should emphasize, all T^* dependence is assumed negligible outside the critical layer, for $|Y^*|$ large. This stringent assumption seems necessary in the present contexts because the T^* dependence is a relatively fast one, so fast that its presence globally, outside the critical layer, would negate the assumed near-neutral form of solutions there and alter the structure set out above (unlike in Stewartson's (1981) geophysical examples, for instance). The global presence of such a fast scale therefore forces a rather more involved unsteady flow, a possibility which we mostly leave aside for now. We note that in (2.9) the case $\mu = 0$ retrieves Stewartson's (1981) equation (see also Hickernell 1984; Stewartson 1978). Although the time-dependent version (2.9) for a moving nonlinear viscous critical layer is clearly important as far as an initial-value problem (and instability) within the critical layer itself is concerned, our interest is as much in the *global* time dependence (and instability) of the flow. This alone, as we shall see, has some significant facets to it. It is therefore tempting to take $\partial/\partial T^* \equiv 0$ below, with unsteadiness then still significant because of the time dependence present in the global and critical-layer flow through the factors μ and γ_c. But the following analysis could be modified ultimately to include the faster $\partial/\partial T^*$ effect, i.e. to consider the dynamics both inside and outside the critical layer, with or without the rather strict assumption above about the extent of the T^* dependence in the present contexts: see an example in §5.1.

3. The boundary layer, channel flow and liquid-layer flow

With the general arguments for time-dependent moving nonlinear critical layers established in §2, we turn to three specific examples of wide concern: the boundary layer, channel flow and liquid-layer flow. The main novel parts of the nonlinear non-neutral disturbance structures for the three basic flows are set out in §§3.1–3.3. An extra common feature here is that the disturbance amplitude must usually vary also on temporal and spatial scales that, although still slow, are faster than the time- and space-scales of the critical-layer movement described in §2. These extra scales turn out to be most significant with regard to the nonlinear development of the disturbance from an initially small state.

3.1. *The boundary layer*

In the five-zoned account (Smith & Bodonyi 1982*a*) of the disturbance structure necessary here, the major changes due to unsteadiness arise near the wall, in the thin predominantly inviscid zone (IZ) surrounding the critical layer and within the thinner nonlinear critical layer (CL) itself (see figure 1*a*). In IZ we have now the expansions, implied mainly by the time- and space-scales introduced above, with $\sigma_* = \epsilon^6 = Re^{-\frac{1}{2}}$, $\alpha_* = \epsilon^{-5}$, and $c_* = \epsilon$,

$$u = \epsilon\lambda \tilde{Y} + \epsilon^2\lambda_2 \tilde{Y}^2 + \epsilon^{\frac{7}{3}}\bar{u}^{(0)} + \epsilon^{\frac{8}{3}}u_{\mathrm{MF}} + \epsilon^{\frac{10}{3}}\bar{u}^{(1)} + \epsilon^{\frac{11}{3}}\bar{u}^{(2)} + \ldots, \tag{3.1a}$$

$$v = \epsilon^{\frac{13}{3}}\bar{v}^{(0)} + \epsilon^{\frac{16}{3}}\bar{v}^{(1)} + \epsilon^{\frac{17}{3}}\bar{v}^{(2)} + \ldots, \tag{3.1b}$$

$$p = \epsilon^{\frac{10}{3}}\bar{p}^{(0)} + \epsilon^{\frac{13}{3}}\bar{p}^{(1)} + \epsilon^{\frac{14}{3}}\bar{p}^{(2)} + \ldots. \tag{3.1c}$$

Here the mean-flow correction u_{MF} is independent of $\tilde{x}$, and the constants λ and λ_2 are the skin friction and curvature of the basic boundary layer near the wall, $\lambda > 0$, $\lambda_2 < 0$, for an inflection-free boundary layer under a favourable pressure gradient, while $\epsilon^{12} = Re^{-1} \ll 1$ and $(x, y, t) = (\epsilon^5\tilde{x}, \epsilon^7\tilde{Y}, \epsilon^4\tilde{t})$ with $\tilde{x}$, $\tilde{Y}$ and $\tilde{t}$ of order unity. For other notation see Smith & Bodonyi (1982*a*). The slower spatial and temporal scales $x = \epsilon^5(\epsilon^{-1}X_1) = \epsilon^5(\epsilon^{-\frac{4}{3}}X_2)$ and $t = \epsilon^4(\epsilon^{-1}T_1) = \epsilon^4(\epsilon^{-\frac{4}{3}}T_2)$ are also present, so that

$$\frac{\partial}{\partial x} \to \epsilon^{-5}\left(\frac{\partial}{\partial \tilde{x}} + \epsilon\frac{\partial}{\partial X_1} + \epsilon^{\frac{4}{3}}\frac{\partial}{\partial X_2}\right), \tag{3.2a}$$

$$\frac{\partial}{\partial t} \to \epsilon^{-4}\left(-c_0\frac{\partial}{\partial \tilde{x}} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^{\frac{4}{3}}\frac{\partial}{\partial T_2}\right), \tag{3.2b}$$

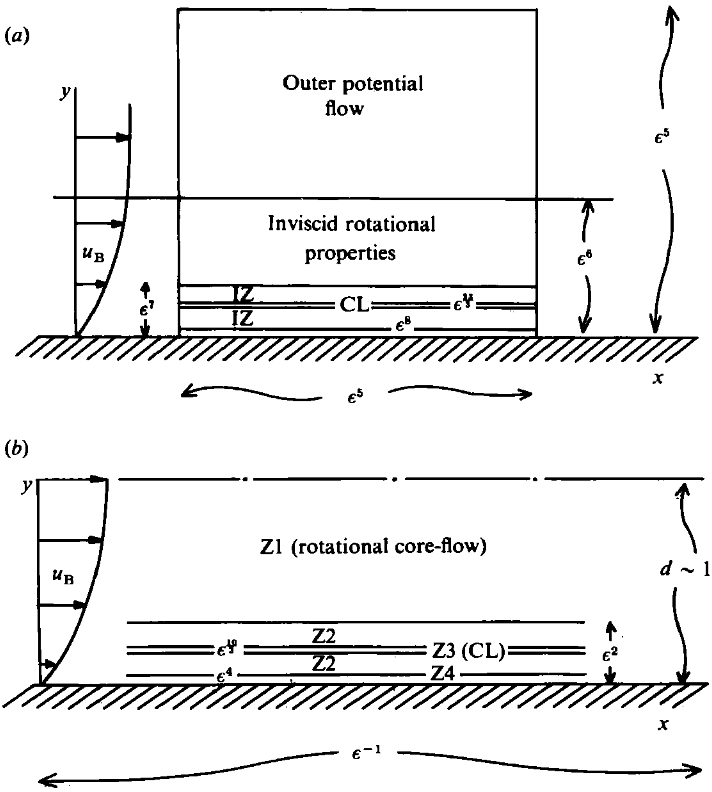

FIGURE 1. Diagrams of the nonlinearly disturbed flow structure: (*a*) for an attached boundary layer as described in §3.1 (see also Bodonyi *et al.* 1983); (*b*) for channel flow as in §3.2, for a symmetric basic motion. In (*a*) $Re = \epsilon^{-12}$ and in (*b*) $Re = \epsilon^{-11}$.

for an underlying but varying wavespeed c_0. Hence the fundamental disturbance in (3.1*a–c*) has the form

$$\bar{u}^{(0)} = 2\lambda A_0 \cos\theta, \quad \bar{v}^{(0)} = 2\alpha_0 \lambda A_0 \bar{Y} \sin\theta, \quad \bar{p}^{(0)} = 2P_0 \cos\theta, \tag{3.3}$$

satisfying the tangential flow condition as the wall is approached, with A_0 and P_0 dependent on X_1, T_1, X_2 and T_2, and (like the effective wavenumber α_0 and wavespeed $c_0 = \beta_0 \alpha_0^{-1}$) unknown. It is necessary for consistency, however, that α_0, c_0 and hence the frequency $\beta_0 = \alpha_0 c_0$ remain independent of X_1 and T_1, depending instead on X_2 and T_2 principally. In other words, physically, a slow variation in α_0, c_0 and β_0, corresponding to a slow movement of the critical layer on the X_2 and T_2 scales, tends to force a much faster-scale (X_1, T_1) response in the global behaviour of the disturbance amplitude A_0 itself. In (3.3) the argument θ satisfies

$$\frac{\partial\theta}{\partial\tilde{x}} = \alpha_0(X_2, T_2), \quad \frac{\partial\theta}{\partial\tilde{t}} = -\beta_0(X_2, T_2), \tag{3.4a}$$

to our order of working, and so we have a *distorted* travelling wave. For compatibility (3.4*a*) then requires that α_0 and β_0 must be related by

$$\frac{\partial\alpha_0}{\partial T_2} = -\frac{\partial\beta_0}{\partial X_2}. \tag{3.4b}$$

In addition the streamwise momentum balance yields the requirement

$$\lambda A_0 \beta_0 = \alpha_0 P_0, \tag{3.4c}$$

because of (3.3); while the potential flow holding outside the boundary layer forces the classical pressure–displacement relation

$$P_0 = \alpha_0 A_0 \tag{3.4d}$$

to hold. So from (3.4*b–d*) we have the equations

$$\left.\begin{aligned}\left[\frac{\partial}{\partial T_2}+2c_0(X_2,T_2)\frac{\partial}{\partial X_2}\right]\alpha_0 = \left[\frac{\partial}{\partial T_2}+2c_0(X_2,T_2)\frac{\partial}{\partial X_2}\right]c_0 = 0,\\ \text{with} \qquad \alpha_0 = \lambda c_0,\end{aligned}\right\} \tag{3.4e}$$

governing the slow variation of α_0 and c_0 in time and space. As expected, the group velocity $c_g = \partial\beta_0/\partial\alpha_0 = 2c_0$ controls the group propagation, at *twice* the local wavespeed.

It is significant also that if disturbances of *fixed* frequency β_0 were under consideration this would correspond immediately to a fixed wavenumber α_0 and a fixed wavespeed c_0, since $\alpha_0 = \beta_0^{\frac{1}{2}}\lambda^{\frac{1}{2}}$ and $c_0 = \beta_0^{\frac{1}{2}}\lambda^{-\frac{1}{2}}$ from (3.4*e*). Our main concern here, however, is with non-uniform conditions of frequency, wavespeed and wavenumber.

At the next order, from the Navier–Stokes equations, (3.1*a–c*) yield the solution for $\bar{v}^{(1)}$:

$$\begin{aligned}\bar{v}^{(1)} = &-\lambda^{-1}\left(\frac{\partial\bar{p}^{(1)}}{\partial\tilde{x}}+\frac{\partial\bar{p}^{(0)}}{\partial X_1}\right)-\lambda\xi\frac{\partial\hat{a}}{\partial\tilde{x}}\\ &+2\alpha_0\lambda_2 A_0(\xi^2+2\lambda^{-1}c_0\xi\ln\xi-\lambda^{-2}c_0^2)\sin\theta-\lambda^{-1}\left(\frac{\partial}{\partial T_1}+c_0\frac{\partial}{\partial X_1}\right)(\bar{u}^{(0)}),\end{aligned} \tag{3.5a}$$

with

$$\partial\bar{p}^{(1)}/\partial\tilde{Y} = 0, \tag{3.5b}$$

the function $\hat{a}$ being independent of $\tilde{Y}$ and as in Smith & Bodonyi (1982*a*) and $\xi \equiv \tilde{Y}-\lambda^{-1}c_0$. Here the influences of the profile curvature λ_2 and of the first slow scales X_1 and T_1 first take effect; the scales are selected for that reason. Also the logarithm in (3.5*a*) is appropriate to $\tilde{Y} > \lambda^{-1}c_0$ above the critical layer. Beneath it the replacement

$$(\ln\xi)\sin\theta \rightarrow (\ln|\xi|)\sin\theta+\phi\cos\theta \quad (\tilde{Y} < \lambda^{-1}c_0) \tag{3.5c}$$

is necessary in view of the (unknown) phase shift ϕ across the critical layer (see below) at $\tilde{Y} \approx \lambda^{-1}c_0$ together with a replacement for the function $\hat{a}$ akin to that in Smith & Bodonyi (1982*a*). So the inviscid displacement effect produced near the wall as $\tilde{Y}\rightarrow 0+$ is $O(\epsilon^{\frac{16}{3}})$, given by $\bar{v}^{(1)}(\tilde{Y}=0+)$. This matches the classical viscous wall-layer displacement

$$\bar{v}(\tilde{Y}=0+) = \epsilon^{\frac{16}{3}}\left[-\frac{2\alpha_0^{\frac{1}{2}}}{c_0^{\frac{3}{2}}}P_0\sin(\theta+\tfrac{1}{4}\pi)+O(\epsilon)\right], \tag{3.6}$$

provided that, from the $\cos\theta$ components of (3.5*a–c*) and (3.6), the phase balance

$$-2\left[\frac{1}{\lambda}\frac{\partial P_0}{\partial X_1}+\left(\frac{\partial}{\partial T_1}+c_0\frac{\partial}{\partial X_1}\right)A_0\right] = 4\alpha_0\lambda_2\frac{c_0^2}{\lambda^2}A_0\phi-\frac{2^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{c_0^{\frac{3}{2}}}P_0 \tag{3.7}$$

holds; the other components play a passive role at this stage. Equation (3.7) then leads ultimately to the nonlinear amplitude equation for A_0. Note that in a neutral state the left-hand side of (3.7) is zero and the right-hand side fixes the steady neutral amplitude via the phase shift. Again, for infinitesimal disturbances, where $\phi = -\pi$, (3.7) determines the small growth rate holding across the majority of the neutral curve at large Reynolds numbers.

Similar reasoning applies to the first appearance of the still slower scales X_2 and T_2, in the forced solution

$$\bar{v}^{(2)} = -\frac{1}{\lambda}\left[\frac{\partial \bar{p}^{(0)}}{\partial X_2} + \left(\frac{\partial}{\partial T_2} + c_0 \frac{\partial}{\partial X_2}\right)\bar{u}^{(0)}\right] - \lambda \xi \frac{\partial B}{\partial \tilde{x}} \tag{3.8a}$$

in IZ, with B independent of $\tilde{Y}$. There are two other sources of forcing invoked at this stage. One is from the pressure–displacement interaction, which requires that

$$B = \frac{P_0}{\alpha_0^3}\frac{\partial \alpha_0}{\partial X_2}\sin\theta.$$

The other is from a phase shift proportional to ϕ_2 in B across CL: see below. There is no corresponding viscous displacement from the wall layer, however, from (3.6), so that the constraint $\bar{v}^{(2)}(\tilde{Y} = 0+) = 0$ holds. Consequently (3.8a) leads to the result

$$\frac{1}{\lambda}\frac{\partial}{\partial X_2}(\alpha_0 A_0) + \left(\frac{\partial}{\partial T_2} + c_0\frac{\partial}{\partial X_2}\right)A_0 + \tfrac{1}{2}c_0\frac{A_0}{\alpha_0}\frac{\partial \alpha_0}{\partial X_2} \propto \phi_2, \tag{3.8b}$$

in view of (3.4d).

The slower scales X_2 and T_2 therefore seem to play a fairly minor role in IZ, at first sight. Their importance is enhanced within the time-dependent critical layer CL, by contrast. In CL, where $\tilde{Y} = \lambda^{-1}c_0 + \epsilon^{\frac{2}{3}}Y$ with Y of order unity and the flowspeed and effective wavespeed nearly coincide, the flow solution has the form

$$\begin{aligned} u = {} & \epsilon c_0 + \epsilon^{\frac{5}{3}}\lambda Y + \epsilon^2 C^{(1)} + 2\epsilon^{\frac{7}{3}}\lambda A_0 \cos\theta \\ & + \epsilon^{\frac{8}{3}}C^{(2)}Y + \epsilon^{\frac{10}{3}}(\ln\epsilon)\,C^{(3)}A_0\cos\theta + \epsilon^{\frac{10}{3}}U_1 + \epsilon^{\frac{11}{3}}U_2 + \dots, \end{aligned} \tag{3.9a}$$

$$\begin{aligned} v = {} & 2\epsilon^{\frac{13}{3}}\alpha_0 A_0 c_0 \sin\theta + 2\epsilon^5\alpha_0\lambda A_0\, Y\sin\theta \\ & + \epsilon^{\frac{16}{3}}C^{(4)} + \epsilon^{\frac{17}{3}}C^{(5)} + \epsilon^6(\ln\epsilon)\,C^{(6)}Y\sin\theta + \epsilon^6 V_1 + \epsilon^{\frac{19}{3}}V_2 + \dots, \end{aligned} \tag{3.9b}$$

$$p = 2\epsilon^{\frac{10}{3}}P_0\cos\theta + \epsilon^{\frac{13}{3}}C^{(7)} + \epsilon^{\frac{14}{3}}C^{(8)} + \epsilon^5 P_1 + \epsilon^{\frac{16}{3}}P_2 + \dots, \tag{3.9c}$$

where $C^{(n)}, n \geqslant 1$, are functions independent of Y and are determined directly by matching with the solution in IZ. For example, $C^{(1)} = \lambda_2 c_0^2 \lambda^{-2}$ and $C^{(2)} = 2\lambda_2 c_0 \lambda^{-1}$. More significantly, since the critical layer is not stationary here, the multiple scaling becomes

$$\frac{\partial}{\partial x} \to \epsilon^{-5}\left[\frac{\partial}{\partial \tilde{x}} + \epsilon\frac{\partial}{\partial X_1} + \epsilon^{\frac{4}{3}}\left(-\frac{\epsilon^{-\frac{2}{3}}}{\lambda}\frac{\partial c_0}{\partial X_2}\frac{\partial}{\partial Y} + \frac{\partial}{\partial X_2}\right) + \dots\right], \tag{3.10a}$$

$$\frac{\partial}{\partial t} \to \epsilon^{-4}\left[-c_0\frac{\partial}{\partial \tilde{x}} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^{\frac{4}{3}}\left(-\frac{\epsilon^{-\frac{2}{3}}}{\lambda}\frac{\partial c_0}{\partial T_2}\frac{\partial}{\partial Y} + \frac{\partial}{\partial T_2}\right) + \dots\right], \tag{3.10b}$$

together with $\partial/\partial y \to \epsilon^{-\frac{23}{3}}\partial/\partial Y$. So, as noted before, time dependence is dominated by the induced Y-derivative in CL.

In CL the Navier–Stokes equations subject to (3.9a–c) and (3.10a,b) reduce to the controlling equations for U_1, V_1 and P_1:

$$\frac{\partial U_1}{\partial \tilde{x}} + 2\lambda\frac{\partial A_0}{\partial X_1}\cos\theta + \frac{\partial V_1}{\partial Y} = 0, \tag{3.11a}$$

$$\begin{aligned} -\left(\frac{\partial \kappa}{\partial T_2} + c_0\frac{\partial \kappa}{\partial X_2}\right)\left(\frac{\partial U_1}{\partial Y} - 2\lambda_2 Y\right) + \lambda Y\left(\frac{\partial U_1}{\partial \tilde{x}} + 2\lambda\frac{\partial A_0}{\partial X_1}\cos\theta\right) & \\ + 2\alpha_0 A_0 c_0 \sin\theta\frac{\partial U_1}{\partial Y} + \lambda V_1 = -\frac{\partial P_1}{\partial \tilde{x}} + \frac{\partial^2 U_1}{\partial Y^2}&, \end{aligned} \tag{3.11b}$$

$$0 = -\frac{\partial P_1}{\partial Y}. \tag{3.11c}$$

Here $\kappa \equiv \lambda^{-1}c_0$ fixes the critical-layer position and is an unknown function of X_2 and T_2. Hence the vorticity equation (2.8) is obtained, from the Y-derivative of (3.11*b*) with (3.11*a*), with X^*, ζ^* and Y^* in (2.8) defined by

$$X^* \equiv \theta, \quad \zeta^* \equiv (2\kappa A_0)^{-\frac{1}{2}}(2\lambda_2)^{-1}\frac{\partial U_1}{\partial Y}, \quad Y^* \equiv (2\kappa A_0)^{-\frac{1}{4}}\, Y \tag{3.12a}$$

and

$$\mu \equiv \frac{\partial c_0/\partial T_2 + c_0\,\partial c_0/\partial X_2}{2\lambda\alpha_0 c_0 A_0}, \quad \gamma_c \equiv \lambda^{\frac{1}{2}}(2c_0 A_0)^{-\frac{3}{4}}\alpha_0^{-1} \tag{3.12b}$$

in the present case. We need also the next-order terms within CL, however, to complete the determination of the disturbance. These are governed by the equation

$$-\left(\frac{\partial\kappa}{\partial T_2}+c_0\frac{\partial\kappa}{\partial X_2}\right)\frac{\partial\zeta_2}{\partial Y}+\lambda Y\frac{\partial\zeta_2}{\partial\tilde{x}}+2\alpha_0 A_0 c_0\frac{\partial\zeta_2}{\partial Y}\sin\theta-\frac{\partial^2\zeta_2}{\partial Y^2} = -\left[\left(\frac{\partial}{\partial T_1}+c_0\frac{\partial}{\partial X_1}\right)\frac{\partial U_1}{\partial Y}+C^{(1)}\frac{\partial^2 U_1}{\partial\tilde{x}\,\partial Y}\right] \tag{3.12c}$$

for the vorticity contribution $\zeta_2 \equiv \partial U_2/\partial Y$. In normalized form this becomes

$$\mu\frac{\partial\zeta_2^*}{\partial Y^*}+Y^*\frac{\partial\zeta_2^*}{\partial X^*}+\frac{\partial\zeta_2^*}{\partial Y^*}\sin X^*-\gamma_c\frac{\partial^2\zeta_2^*}{\partial Y^{*2}} = -\frac{1}{\mu_2}\left[\gamma_2\left(\frac{\partial}{\partial T_1}+c_0\frac{\partial}{\partial X_1}\right)+\frac{\partial}{\partial X^*}\right](\mu_2\zeta^*), \tag{3.12d}$$

where $\zeta_2 = 2\lambda_2^{\frac{3}{2}}c_0^2\lambda^{-3}\zeta_2^*, \gamma_2 = \lambda^2\lambda_2^{-1}c_0^{-2}\alpha_0^{-1}$ and $\mu_2 = 2\lambda_2(2\kappa A_0)^{\frac{1}{2}}$. The matching condition here is

$$\zeta_2^* \sim O\left(\frac{1}{Y^*}\right) \quad \text{as } Y^* \to \pm\infty, \tag{3.12e}$$

and the solution of (3.12*d, e*) is required to fix the secondary phase shift ϕ_2, where ϕ_2 is the jump in the velocity, i.e. in the integral of ζ_2^* with respect to Y^*, as far as the sin θ component only is concerned.

The phase shift ϕ is determined by the solution of (2.8) with the asymptotic constants involving an unknown vorticity jump $2(H^+ - H^-)$,

$$\zeta^* \sim Y^* + 2H^\pm + O(Y^{*-1}), \tag{3.13a}$$

followed by integration to give

$$U^* \sim \tfrac{1}{2}Y^{*2} + 2H^\pm Y^* + \cos\theta \ln|Y^*| + U_\pm^*, \tag{3.13b}$$

where $\zeta^* = \partial U^*/\partial Y^*$. Here ϕ is the $\sin\theta$ component of the velocity jump:

$$\phi = \pi^{-1}\int_0^{2\pi}(U_+^* - U_-^*)\sin\theta\,\mathrm{d}\theta. \tag{3.14}$$

Thus ϕ (and hence ϕ_2) depends on the values of μ and γ_c, which act as quasi-steady parameters in the governing equation (2.8) itself (and in (3.12*d*)). With $\phi(\mu, \gamma_c)$ and $\phi_2(\mu, \gamma_c)$ determined, the spatial and temporal growth or decay of the amplitude A_0 follows from (3.7) with (3.4*c, d*), i.e.

$$\left(\frac{\partial}{\partial T_1}+2c_0\frac{\partial}{\partial X_1}\right)A_0 = -2\frac{\lambda_2}{\lambda}c_0^3 A_0[\phi(\mu,\gamma_c)-\phi_s], \tag{3.15a}$$

where the negative constant $\phi_s = -\lambda^{\frac{5}{2}}2^{-\frac{3}{2}}(-\lambda_2)^{-1}c_s^{-3}$ is the value of the phase shift ϕ in a neutral steady state with quasi-wavespeed $c_0 \to c_s$ (constant): whether such a state is achieved or not remains to be seen. The nonlinearity in (3.15*a*) is due in particular to the dependence of γ_c on A_0 (see e.g. §4 below). Unsteadiness appears explicitly in the left-hand side of (3.15*a*) and implicitly through μ in the right-hand side. In addition to (3.15*a*) the slower scale behaviour of $A_0(X_1, T_1, X_2, T_2)$ is governed by

$$\frac{\partial}{\partial X_2}(c_0 A_0)+\left(\frac{\partial}{\partial T_2}+c_0\frac{\partial}{\partial X_2}\right)A_0 \propto \phi_2, \tag{3.15b}$$

from (3.8*b*), and by the slow-scale response of $c_0(X_2, T_2)$ given by

$$\left(\frac{\partial}{\partial T_2}+2c_0\frac{\partial}{\partial X_2}\right)c_0 = 0, \tag{3.15c}$$

from (3.4*e*). In (3.15*a*) a steady neutral state, *if* it is attainable (see §4), is achieved when the spatial and temporal dependence decay to zero, i.e. $(\partial/\partial X_1, \partial/\partial T_1, \partial/\partial X_2, \partial/\partial T_2)\rightarrow 0$: then $\mu \rightarrow 0$ and $c_0 \rightarrow c_S$.

The terms in the left-hand side of (3.15*a*) are identical to those obtained by Benney & Maslowe (1975) in their study of slowly varying nonlinear waves in the presence of critical layers (see also Stewartson & Stuart 1971).

The above arguments may possibly be generalized to recover (2.9) with the scalings discussed in §2, at least provided the T^* dependence can be suppressed outside the critical layer, as pointed out earlier. See also §5.1.

3.2. *Channel flow*

The structure of a disturbed flow with a nonlinear critical layer here involves primarily the long lengthscale $x = O(Re^{\frac{1}{11}})$ along the channel and four zones (Z1–Z4) laterally, the channel being given by $-d \leqslant y \leqslant d$ for all x and the basic channel flow by a general symmetric profile $\bar{u} = u_B(y)$ with $\bar{v} = 0$, $\bar{p}+\text{constant} \propto Re^{-1}x$ in effect (see figure 1*b*). Here $u_B(y) \sim \lambda(y+d)+\lambda_2(y+d)^2$ near the lower wall $y = -d$, with λ positive but λ_2 negative. An example is plane Poiseuille flow.

Zone Z1 occupies the majority of the channel $-d < y < d$, and the flow solution there has the form

$$u = u_B(y)+\epsilon^{\frac{14}{3}}u_0+\epsilon^{\frac{16}{3}}u_{MF}+\epsilon^{\frac{20}{3}}u_1+\epsilon^{\frac{22}{3}}u_2+\ldots, \tag{3.16a}$$

$$v = \epsilon^{\frac{17}{3}}v_0+\epsilon^{\frac{23}{3}}v_1+\epsilon^{\frac{25}{3}}v_2+\ldots, \tag{3.16b}$$

$$p = \epsilon^{\frac{20}{3}}p_0+\epsilon^{\frac{26}{3}}p_1+\epsilon^{\frac{28}{3}}p_2+\ldots. \tag{3.16c}$$

In addition, the multiple scales

$$\frac{\partial}{\partial x}\rightarrow \epsilon\alpha_0\frac{\partial}{\partial X}+\epsilon^3\frac{\partial}{\partial X_1}+\epsilon^{\frac{11}{3}}\frac{\partial}{\partial X_2}+\ldots, \tag{3.17a}$$

$$\frac{\partial}{\partial t}\rightarrow -\epsilon^3\alpha_0 c_0\frac{\partial}{\partial X}+\epsilon^5\frac{\partial}{\partial T_1}+\epsilon^{\frac{17}{3}}\frac{\partial}{\partial T_2}+\ldots \tag{3.17b}$$

apply; and now $\epsilon^{11} \equiv Re^{-1} \ll 1$. Also, $\sigma_* = 1$, $\alpha_* = \epsilon$ and $c_* = \epsilon^2$. In (3.16*a*) u_{MF} is independent of X. The Navier–Stokes equations then yield the successive solutions

$$u_0 = A_0 u_B(y)\,C, \quad v_0 = \alpha_0 A_0 u_B(y)\,S, \tag{3.18a,b}$$

$$p_0 = p_{00}-C\alpha_0^2 A_0\int_{-d}^{y} u_B^2(y)\,\mathrm{d}y \tag{3.18c}$$

and $$v_1 = \alpha_0 u_B \int_0^y \frac{\partial p_0}{\partial X}\frac{dy}{u_B^2} - \alpha_0 c_0 A_0 S + \alpha_0 u_B \frac{\partial \hat{g}}{\partial X}, \tag{3.18d}$$

$$p_1 = p_{10} - \int_{-d}^{y} u_B^2 \frac{\partial^2}{\partial X^2}\left\{\int_0^y \alpha_0^2 \frac{p_0\, dy}{u_B^2}\right\} dy + 2\alpha_0^2 c_0 A_0 C \int_{-d}^{y} u_B\, dy - \left\{\alpha_0^2 \frac{\partial^2 \hat{g}}{\partial X^2} + \alpha_0 \frac{\partial A_0}{\partial X_1} S\right\} \int_{-d}^{y} u_B^2\, dy \tag{3.18e}$$

and $$v_2 = -\alpha_0 \frac{\partial G}{\partial X} u_B, \tag{3.18f}$$

$$p_2 = p_{20} + \left\{\alpha_0^2 \frac{\partial^2 G}{\partial X^2} - \frac{\partial(\alpha_0 A_0)}{\partial X_2} S\right\} \int_{-d}^{y} u_B^2\, dy, \tag{3.18g}$$

where $S \equiv \sin X$, $C \equiv \cos X$. Here, as in §3.1, the fundamental wave, in (3.18*a*–*c*), is *non-simple* since α_0 and c_0 vary with both X_2 and T_2. The other unknown functions arising, A_0, p_{00}, $\hat{g}$, p_{10}, G and p_{20}, depend on the faster scales X_1 and T_1 as well as X_2 and T_2, and all but A_0 also vary with X. In the solution (3.18*a*–*c*) $-A_0$ is the inviscid displacement effect, while (3.18*c*) reflects the importance of the cross-channel pressure gradient produced by the curvature of that displacement. The time- and space-scales X_2 and T_2 above, which govern the critical-layer movement, are inferred from the general argument of §2, since in the present case c in (2.3) is $O(\epsilon^2)$. The faster scales X_1 and T_1 on which the typical disturbance amplitude A_0 then reacts are induced in response to the traditional logarithmic singularity, below, due to the curvature λ_2 of the basic flow.

Next, zone Z2 near the lower wall $y = -d$ is a continuation of Z1, but it details the position of the critical layer (Z3). In Z2, $y = -d + \epsilon^2 \tilde{Y}$, with $\tilde{Y} = O(1)$, and the expansions

$$u = \epsilon^2 \lambda \tilde{Y} + \lambda_2 \epsilon^4 \tilde{Y}^2 + \epsilon^{\frac{14}{3}} \bar{u}_0 + \epsilon^{\frac{16}{3}} \bar{u}_{MF} + \epsilon^{\frac{20}{3}} \bar{u}_1 + \epsilon^{\frac{22}{3}} \bar{u}_2 + \ldots, \tag{3.19a}$$

$$v = \epsilon^{\frac{23}{3}} \bar{v}_0 + \epsilon^{\frac{29}{3}} \bar{v}_1 + \epsilon^{\frac{31}{3}} \bar{v}_2 + \ldots, \tag{3.19b}$$

$$p = \epsilon^{\frac{20}{3}} \bar{p}_0 + \epsilon^{\frac{26}{3}} \bar{p}_1 + \epsilon^{\frac{28}{3}} \bar{p}_2 + \ldots \tag{3.19c}$$

hold, with $\bar{u}_{MF}$ independent of x. Coupled with (3.17*a*,*b*) still, (3.19*a*–*c*) yield the solutions

$$\bar{u}_0 = A_0 \lambda C, \quad \bar{v}_0 = \lambda \tilde{Y} A_0 \alpha_0 S, \quad \frac{\partial \bar{p}_0}{\partial X} = -c_0 A_0 \lambda S \tag{3.20a}$$

at leading order. These match with the forms in $Z1$ and give tangential flow at the wall ($\tilde{Y} \to 0+$) as required. Next we obtain

$$\bar{v}_1 = -\frac{1}{\lambda}\left(\alpha_0 \frac{\partial \bar{p}_1}{\partial X} + \frac{\partial \bar{p}_0}{\partial X_1} + \frac{\partial \bar{u}_0}{\partial T_1} + c_0 \frac{\partial \bar{u}_0}{\partial X_1}\right) - 2\lambda_2 \frac{\alpha_0 A_0 S}{\lambda^2}\left(\frac{c_0^2}{2} - c_0 \xi \ln \xi - \frac{\xi^2}{2}\right) - \xi \alpha_0 \frac{\partial \hat{a}}{\partial X}, \tag{3.20b}$$

followed by

$$\bar{v}_2 = -\frac{1}{\lambda}\left(\alpha_0 \frac{\partial \bar{p}_2}{\partial X} + \frac{\partial \bar{p}_0}{\partial X_2} + \frac{\partial \bar{u}_0}{\partial T_2} + c_0 \frac{\partial \bar{u}_0}{\partial X_2} + \alpha_0 \bar{u}_0 \frac{\partial \bar{u}_0}{\partial X}\right) - \xi \alpha_0 \frac{\partial B}{\partial X}, \tag{3.20c}$$

again from the Navier–Stokes equations. Here $\xi \equiv \lambda \tilde{Y} - c_0$, $\bar{p}_0, \bar{p}_1$ and $\bar{p}_2$ are independent of $\tilde{Y}$, and $\hat{a}$ and B are further unknown displacement functions of

$X, X_1, T_1, X_2, T_2, \ldots$. Beneath the critical layer for $\xi < 0$ the replacement (3.5*c*) deals with the logarithmic contributions in (3.20*b*), with ϕ again unknown and S and C replacing $\sin\theta$ and $\cos\theta$ now. There are also, as in §3.1, non-monochromatic jumps in the functions $\hat{a}$ and B due to Z3 but not involving any C-contributions in $\hat{a}$.

Before addressing the nonlinear time-dependent critical layer Z3 we note that the $O(\epsilon^4)$-thick viscous wall layer Z4 remains of classical linear form, and so produces the viscous displacement condition

$$\bar{v}(\bar{Y}\to 0+) = \epsilon^{\frac{29}{3}}\left[-\lambda A_0\left(\frac{\alpha_0}{c_0}\right)^{\frac{1}{2}}\sin(X+\tfrac{1}{4}\pi)+O(\epsilon^2)\right]. \quad (3.21a)$$

Therefore the join between Z2 and Z4 requires the two relations

$$-\frac{1}{\lambda}\left(\alpha_0\tilde{p}_{11}+\frac{\partial\tilde{p}_{00}}{\partial X_1}\right)-\left(\frac{\partial A_0}{\partial T_1}+c_0\frac{\partial A_0}{\partial X_1}\right)+\alpha_0 c_0\left[\tilde{a}_1-2\lambda_2\frac{c_0 A_0}{\lambda^2}\phi\right] = -\lambda A_0\left(\frac{\alpha_0}{2c_0}\right)^{\frac{1}{2}}, \quad (3.21b)$$

$$-\frac{1}{\lambda}\left(\alpha_0\tilde{p}_{21}+\frac{\partial\tilde{p}_{00}}{\partial X_2}\right)-\left(\frac{\partial A_0}{\partial T_2}+c_0\frac{\partial A_0}{\partial X_2}\right)+\alpha_0 c_0[\tilde{B}_1-\phi_2] = 0 \quad (3.21c)$$

to be satisfied, as regards the all-important C-contributions. In the above, $\tilde{p}_{00}$ and $\tilde{p}_{11}$ are the C-components of $\bar{p}_0$ and $\bar{p}_1$ respectively, and $\tilde{p}_{21}, \tilde{a}_1$ and $\tilde{B}_1$ are the S-components of the functions $\bar{p}_2, \hat{a}$ and B above Z3; in fact $\bar{p}_0 = \tilde{p}_{00}C$, and the x-momentum balance in Z2 gives the relation

$$(C^{-1}p_{00} =)\,\tilde{p}_{00} = \lambda c_0 A_0. \quad (3.21d)$$

Below the critical layer Z3, $\tilde{a}_1$ is unchanged but $\tilde{B}_1$ suffers a jump, given by ϕ_2, to be determined from the critical-layer properties.

Along with (3.21*b*–*d*), the relations

$$\tilde{p}_{00} = \alpha_0^2 A_0 I_1, \quad \tilde{p}_{11} = \left(\alpha_0^2\tilde{a}_1+\alpha_0\frac{\partial A_0}{\partial X_1}\right)I_1, \quad \tilde{p}_{21} = \left(\alpha_0^2\tilde{B}_1+\frac{\partial(\alpha_0 A_0)}{\partial X_2}\right)I_1 \quad (3.22a,b,c)$$

also apply from the merging between Z2 and Z1, where

$$I_1 \equiv \int_{-d}^{0} u_B^2(y)\,dy \quad (3.22d)$$

is a given positive constant. The required condition of antisymmetry about the channel centreline $y = 0$ has been used in Z1 for the disturbance pressure. Again, compatibility ($\partial^2/\partial x\,\partial t = \partial^2/\partial t\,\partial x$) requires that $\partial\alpha_0/\partial T_2+\partial(\alpha_0 c_0)/\partial X_2 = 0$. So, combining this with (3.21*d*) and (3.22*a*), we obtain the results

$$c_0 = \frac{\alpha_0^2 I_1}{\lambda}, \quad \frac{\partial\alpha_0}{\partial T_2}+3c_0\frac{\partial\alpha_0}{\partial X_2} = 0 \quad (3.22e)$$

affecting the critical-layer movement. The group velocity $c_g = 3c_0$ is *thrice* the local quasi-wavespeed.

Addressing finally the nonlinear time-dependent viscous critical layer Z3, wherein $y = -d+\epsilon^2\lambda^{-1}c_0+\epsilon^{\frac{10}{3}}Y$, with Y of order unity, we have there the multiple scalings (3.17*a*,*b*), but supplemented by

$$\frac{\partial}{\partial X_2}\to -\epsilon^{-\frac{4}{3}}\lambda^{-1}\frac{\partial c_0}{\partial X_2}\frac{\partial}{\partial Y}+\frac{\partial}{\partial X_2}, \quad (3.23a)$$

$$\frac{\partial}{\partial T_2}\to -\epsilon^{-\frac{4}{3}}\lambda^{-1}\frac{\partial c_0}{\partial T_2}\frac{\partial}{\partial Y}+\frac{\partial}{\partial T_2}, \quad (3.23b)$$

where once again the dependence on critical-layer movement is dominant. The flow solution in Z3 has the form

$$u = \epsilon^2 c_0 + \epsilon^{\frac{10}{3}} \lambda Y + \lambda_2 \lambda^{-2} c_0^2 \epsilon^4 + \epsilon^{\frac{14}{3}} \tilde{u}_0 + \epsilon^{\frac{16}{3}} (\tilde{u}_{\frac{1}{2}} + 2\lambda_2 \lambda^{-1} c_0 Y) + \epsilon^{\frac{20}{3}} (\tilde{u}_1 + \lambda_2 Y^2) + \epsilon^{\frac{22}{3}} \tilde{u}_2 + \dots, \tag{3.24a}$$

$$v = \epsilon^{\frac{23}{3}} \tilde{v}_{-\frac{1}{2}} + \epsilon^{\frac{27}{3}} \tilde{v}_0 + \epsilon^{\frac{29}{3}} \tilde{v}_{\frac{1}{2}} + \epsilon^{\frac{31}{3}} \tilde{v}_{31} + \epsilon^{\frac{33}{3}} \tilde{v}_1 + \epsilon^{\frac{35}{3}} \tilde{v}_2 + \dots, \tag{3.24b}$$

$$p = \epsilon^{\frac{20}{3}} \tilde{p}_0 + \epsilon^{\frac{26}{3}} \tilde{p}_1 + \epsilon_{\frac{28}{3}} \tilde{p}_2 + \dots, \tag{3.24c}$$

where $\tilde{v}_{-\frac{1}{2}} = -\alpha_0 \lambda^{-1} \partial \tilde{p}_0 / \partial X$.

Substitution into the Navier–Stokes equations provides the successive governing equations of continuity and x-momentum

$$\alpha_0 \frac{\partial \tilde{u}_0}{\partial X} + \frac{\partial \tilde{v}_0}{\partial Y} = \alpha_0 \frac{\partial \tilde{u}_1}{\partial X} + \frac{\partial \tilde{u}_0}{\partial X_1} + \frac{\partial \tilde{v}_1}{\partial Y} \tag{3.25a}$$

$$= \alpha_0 \frac{\partial \tilde{u}_2}{\partial X} + \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X_1} + \frac{\partial \tilde{u}_0}{\partial X_2} + \frac{\partial \tilde{v}_2}{\partial Y} \tag{3.25b}$$

$$= 0 \tag{3.25c}$$

and
$$\mathscr{L}(\tilde{u}_0, \tilde{v}_0) = 0, \tag{3.25d}$$

$$\mathscr{L}(\tilde{u}_{\frac{1}{2}}, \tilde{v}_{\frac{1}{2}}) + \left(\frac{\partial \tilde{u}_0}{\partial T_1} + c_0 \frac{\partial \tilde{u}_0}{\partial X_1} \right) - \frac{\alpha_0 \tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_0}{\partial X} - \frac{\tilde{v}_{-\frac{1}{2}} \tilde{g} c_0}{\lambda} = -\alpha_0 \frac{\partial \tilde{p}_1}{\partial X} - \frac{\partial \tilde{p}_0}{\partial X_1}, \tag{3.25e}$$

$$\mathscr{L}(0, \tilde{v}_{31}) + \left(\frac{\partial \tilde{u}_0}{\partial T_2} + c_0 \frac{\partial \tilde{u}_0}{\partial X_2} \right) - \frac{\alpha_0 \tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X} + \left(\alpha_0 \tilde{u}_0 \frac{\partial \tilde{u}_0}{\partial X} + \tilde{v}_0 \frac{\partial \tilde{u}_0}{\partial Y} \right) = -\alpha_0 \frac{\partial \tilde{p}_2}{\partial X} - \frac{\partial \tilde{p}_0}{\partial X_2}, \tag{3.25f}$$

$$\mathscr{L}(\tilde{u}_1, \tilde{v}_1) + \left(\frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial T_2} + c_0 \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X_2} \right) + \lambda Y \frac{\partial \tilde{u}_0}{\partial X_1} - Y \frac{\partial c_0}{\partial X_2} \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial Y} + \alpha_0 \left(\tilde{u}_{\frac{1}{2}} - \frac{\tilde{g} c_0 Y}{\lambda} \right) \frac{\partial \tilde{u}_0}{\partial X} - \tilde{g} Y \tilde{v}_{-\frac{1}{2}} + \tilde{v}_0 \left(\frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial Y} - \frac{\tilde{g} c_0}{\lambda} \right) + \tilde{v}_{\frac{1}{2}} \frac{\partial \tilde{u}_0}{\partial Y} = -\alpha_0 \frac{\partial \tilde{p}_3}{\partial X}, \tag{3.25g}$$

$$\mathscr{L}(\tilde{u}_2, \tilde{v}_2) + \left(\frac{\partial \tilde{u}_1}{\partial T_1} + c_0 \frac{\partial \tilde{u}_1}{\partial X_1} \right) - \frac{\tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_0}{\partial X_1} + \lambda Y \left(\frac{\partial \tilde{u}_0}{\partial X_2} + \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X_1} \right) - \frac{\alpha_0 \tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_1}{\partial X} - \tilde{v}_{\frac{1}{2}} \frac{\tilde{g} c_0}{\lambda} = -\alpha_0 \frac{\partial \tilde{p}_4}{\partial X} - \frac{\partial \tilde{p}_1}{\partial X_1}, \tag{3.25h}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}(u, v) \equiv -\frac{1}{\lambda} \left(\frac{\partial c_0}{\partial T_2} + c_0 \frac{\partial c_0}{\partial X_2} \right) \frac{\partial u}{\partial Y} + \lambda Y \alpha_0 \frac{\partial u}{\partial X} + \tilde{v}_{-\frac{1}{2}} \frac{\partial u}{\partial Y} + \lambda v - \frac{\partial^2 u}{\partial Y^2}, \tag{3.26}$$

and $\tilde{g} = -2\lambda_2$. The y-momentum balances simply give $\partial \tilde{p}_n / \partial Y = 0$ for $0 \leqslant n \leqslant 4$. Hence we obtain

$$\left. \begin{aligned} \tilde{v}_{-\frac{1}{2}} &= \alpha_0 c_0 A_0 S, \quad \tilde{u}_0 = A_0 \lambda C, \quad \tilde{v}_0 = \lambda Y A_0 \alpha_0 S, \\ \lambda \tilde{v}_{\frac{1}{2}} &= -\left(\alpha_0 \frac{\partial \tilde{p}_1}{\partial X} + \frac{\partial \tilde{p}_0}{\partial X_1} + \frac{\partial \tilde{u}_0}{\partial T_1} + c_0 \frac{\partial \tilde{u}_0}{\partial X_1} \right) + \tfrac{1}{2} \lambda^{-1} \tilde{g} c_0^2 A_0 \alpha_0 S, \\ \lambda \tilde{v}_{31} &= -\left(\alpha_0 \frac{\partial \tilde{p}_2}{\partial X} + \frac{\partial \tilde{p}_0}{\partial X_2} + \frac{\partial \tilde{u}_0}{\partial T_2} + c_0 \frac{\partial \tilde{u}_0}{\partial X_2} \right) - \alpha_0 \tilde{u}_0 \frac{\partial \tilde{u}_0}{\partial X}, \end{aligned} \right\} \tag{3.27}$$

and $\tilde{u}_{\frac{1}{2}}$ is independent of X and Y. The first non-simple problem in (3.25) is that for $\tilde{u}_1$ therefore, and takes exactly the form in (2.8) after the transformations

$$\tilde{u}_1 = -\frac{\tilde{g}c_0 A_0}{\lambda} U^*, \quad Y = \left(\frac{A_0 c_0}{\lambda}\right)^{\frac{1}{2}} Y^*, \tag{3.28a,b}$$

$$\frac{\partial \tilde{u}_1}{\partial Y} - \tilde{g}Y = -\tilde{g}\left(\frac{A_0 c_0}{\lambda}\right)^{\frac{1}{2}} \zeta^*, \tag{3.28c}$$

$$X = X^*$$

are introduced in the Y-derivative of (3.25*g*). Here

$$\mu = \left(\frac{\partial c_0}{\partial T_2} + c_0 \frac{\partial c_0}{\partial X_2}\right) \Big/ \lambda A_0 \alpha_0 c_0, \tag{3.28d}$$

and the far-field conditions (3.13*a*,*b*) are retrieved. The second non-simple problem is for $\tilde{u}_2$. In turn it acquires the form (3.12*d*), where now

$$\zeta_2^* \equiv \tfrac{1}{2}\tilde{g}^{-2} c_0^{-2} \lambda^3 \frac{\partial \tilde{u}_2}{\partial Y}, \quad \gamma_2 \equiv -\tfrac{1}{2}\lambda^2 \tilde{g}^{-1} c_0^{-2} \alpha_0^{-1}, \quad \mu_2 = -\tilde{g}\left(\frac{2c_0 A_0}{\lambda}\right)^{\frac{1}{2}}, \tag{3.28e}$$

and the conditions (3.12*e*) again apply here. So ζ^* and ζ_2^* are responsible for the phase shifts ϕ and ϕ_2 respectively across Z3, which appear in (3.21*b*,*c*).

In summary the unsteady nonlinear disturbance amplitude A_0 and the moving critical-layer position $\lambda^{-1}c_0$ are controlled by the equations

$$\frac{\partial A_0}{\partial T_1} + 3c_0 \frac{\partial A_0}{\partial X_1} = \sigma A_0(\phi - \phi_s), \tag{3.29a}$$

$$\frac{\partial A_0}{\partial T_2} + 3c_0 \frac{\partial A_0}{\partial X_2} + \tfrac{3}{2}A_0 \frac{\partial c_0}{\partial X_2} = -\alpha_0 c_0 \phi_2, \tag{3.29b}$$

$$\frac{\partial c_0}{\partial T_2} + 3c_0 \frac{\partial c_0}{\partial X_2} = 0, \tag{3.29c}$$

(cf. the set (3.15*a*–*c*)), where now

$$\sigma \equiv -2\lambda_2 \frac{c_0^2 \alpha_0}{\lambda^2} > 0, \quad \phi_s \equiv \frac{-\lambda \alpha_0}{(2\alpha_0 c_0)^{\frac{1}{2}} \sigma} < 0, \tag{3.30}$$

with α_0 given by (3.22*e*). Further comments are covered by those at the end of §3.1.

3.3. *Liquid-layer flow*

We consider liquid-layer flow over an inclined plane, and take the plane to be inclined at an angle ω^* to the horizontal with $\omega^* = O(Re^{-\frac{2}{11}})$ and $s = 1/Re^{\frac{2}{11}} \tan \omega^* = 0(1)$. The disturbance structure then takes exactly the same form as in §3.2, the only difference being that the free-surface conditions lead to the eigenrelation

$$\lambda c_0 = \alpha_0^2 I_1 - s \tag{3.22e'}$$

instead of (3.22*e*). See Gajjar (1984) for further details on the neutral case, i.e. without a moving critical layer. The fundamental problem (2.8) and the amplitude equations (3.29) remain unaltered.

For steeper planes ($\omega^* \gg Re^{-\frac{2}{11}}$), $s = 0$ effectively in (3.22*e*′).

4. The amplitude equation for fixed-frequency disturbances: the special case $\mu = 0$

If we consider fixed-frequency disturbances, with $\beta_0 = \text{const.}$, then as mentioned previously the eigerelations (3.4*e*) and (3.22*e*) imply that both c_0 and α_0 are also constant and hence $\mu = 0$. The phase shift ϕ occurring in the amplitude equations is then determined by solving the steady Haberman problem (here we address (2.8) rather than (2.9)). The properties of this are well known: $\phi(A)$ is monotonic, with $\phi \to -\pi$ as $A \to 0$ and $\phi \propto -A^{-\frac{3}{2}}$ for $A \gg 1$ (see e.g. Haberman 1972; Smith & Bodonyi 1982*a*). These properties come mainly from a numerical treatment. Qualitatively, however, they may be represented by a model with $\phi(A)$ given by

$$\phi(A) = \frac{-\pi}{1+k_0 A^{\frac{3}{2}}}, \tag{4.1}$$

where k_0 is a positive constant. If we consider A as a function of T_1 and $z = X_1 - T_1 c_g$, where c_g is the group velocity, then (3.15*a*) and (3.29*a*) with (4.1) take the form

$$\frac{\partial A}{\partial T_1} = \sigma A\left[\frac{-\pi}{1+k_0 A^{\frac{3}{2}}} - \phi_s\right]. \tag{4.2}$$

This integrates to give the implicit solution

$$\frac{|A|}{|\phi_s + \pi + \phi_s k_0 A^{\frac{3}{2}}|^{-2\pi/3\phi_s}} = \exp\{-\sigma T_1(\phi_s + \pi)\}\, M(z), \tag{4.3}$$

where $M(z)$ (> 0) is an arbitrary function of integration; the value $\phi_s = -\pi$ corresponds to the linear neutral modes α_{0n} and c_{0n}.

If $\phi_s < -\pi$, i.e. $\alpha_0 < \alpha_{0n}$, then $c_0 < c_{0n}$ and the initially small disturbance lies inside the linear neutral curve. The nonlinear solution is sketched in figure 2(*a*). It has the following behaviours as $T_1 \to \pm\infty$:

$$A \sim \exp(-\sigma T_1(\phi_s + \pi))\,|\phi_s + \pi|^{-2\pi/3\phi_s} \quad \text{as } T_1 \to -\infty, \tag{4.4a}$$

governing its initial response; and

$$A \sim \exp(-\sigma T_1 \phi_s) \quad \text{as } T_1 \to +\infty, \tag{4.4b}$$

which describes its ultimate form. Here (4.4*a*, *b*) imply that a mode that is initially small and linearly unstable will amplify, become nonlinear and then become unbounded at larger times (as $T_1 \to \infty$).

On the other hand, if $\phi_s > -\pi$, corresponding to the linearly stable modes, we have the significant result that there is an unstable subcritical equilibrium amplitude A_s given by $\phi_s = \phi(A_s)$ as $T_1 \to -\infty$ (see figure 2*b*). If the amplitude A is below A_s then the disturbance amplitude decays to zero as time T_1 increases, whereas if A is above A_s then A becomes unbounded, attaining the behaviour (4.4*b*).

The above conclusions happen to agree (qualitatively only) with those from the weakly nonlinear stability theory of Stuart (1960) and Watson (1960) for smaller disturbances.

Finally it is noted again that the above results are for the special case $\mu = 0$ only. For a moving critical layer $\mu \neq 0$, and the phase shift ϕ is related in a much more complicated manner to A, and not necessarily monotonically as in the model problem above. Even a heuristic discussion of the amplitude equations and the implications

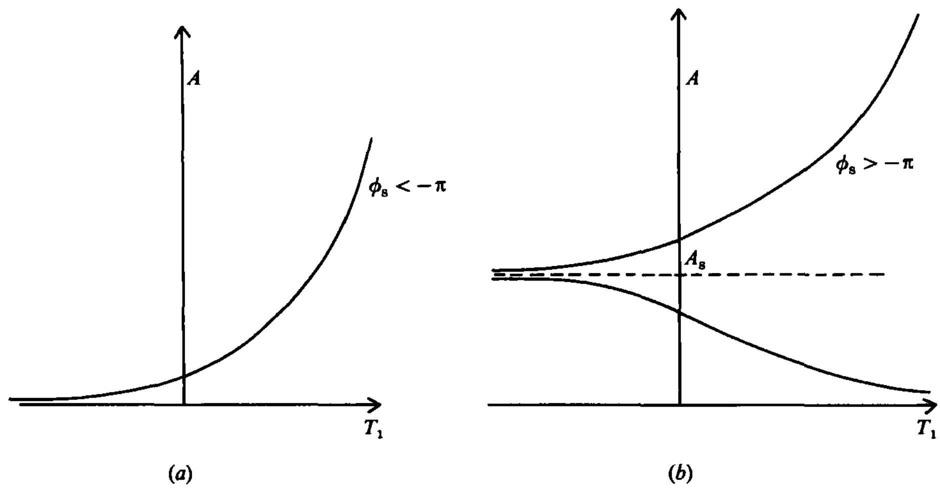

FIGURE 2. Sketch of the nonlinear solutions of the amplitude equations for the special case $\mu = 0$, with $\phi(A) = -\pi/(1+k_0 A^{\frac{3}{2}})$; see §4: (*a*) $\phi_s < -\pi$, corresponding to linearly unstable disturbances; (*b*) $\phi_s > -\pi$, for linearly stable disturbances, showing the nonlinear threshold phenomenon.

for the neutral-stability curve would first require an explicit determination of the functional relation of ϕ on A and this needs a full numerical solution of the problem (2.8).

We therefore turn next to some limiting cases of (2.8).

5. Preliminary properties of the time-dependent nonlinear critical-layer equation

The properties of the nonlinear critical-layer equation when dominated by viscosity, nonlinearity or time dependence are dictated by the various limits of μ and γ_c, large or small, and we devote this section mostly to some preliminaries on these limiting solutions.

First, however, an interesting general property concerning the mean-vorticity jump and the phase shift ϕ is noted. Integrating (2.8) with respect to Y^*, we obtain

$$\mu(Y^*-\zeta^*)+Y^*U^*_{X^*}-\psi^*_{X^*}+\sin X^* U^*_{Y^*} = -P^*_{X^*}+\gamma_c U^*_{Y^*Y^*}, \tag{5.1}$$

with the boundary conditions, for $Y^* \rightarrow \pm\infty$,

$$\psi^* \sim \tfrac{1}{6}Y^{*3}+H^{\pm}Y^{*2}+\cos X^*(Y^*\ln|Y^*|-Y^*)+B^{\pm}(X)\,Y^*+G^{\pm}(X)+O\left(\frac{1}{Y^*}\right), \tag{5.2a}$$

$$U^* \sim \tfrac{1}{2}Y^{*2}+2H^{\pm}Y^*+\cos X^*\ln|Y^*|+B^{\pm}(X)+O\left(\frac{1}{Y^*}\right), \tag{5.2b}$$

$$\zeta^* \sim Y^*+2H^{\pm}+\frac{\cos X^*}{Y^*}+O\left(\frac{1}{Y^3}\right). \tag{5.2c}$$

Integration of (5.1) with respect to X^*, from $X^* = 0$ to $X^* = 2\pi$, and application of the boundary conditions therefore gives

$$-4\mu\pi(H^+-H^-) = 0. \tag{5.3}$$

A further integration of (5.1) with respect to Y^* and then with respect to X^* over a period yields

$$-\mu \int_0^{2\pi} (B^+ - B^-)\,\mathrm{d}X + \int_0^{2\pi} (B^+ - B^-) \sin X\,\mathrm{d}X = 4\pi\gamma_c(H^+ - H^-), \tag{5.4}$$

after use again of (5.2*a*–*c*). Here (5.3) is a notable result, showing that for any non-zero μ there is no vorticity jump across the critical layer. This is in contrast with the steady equation, where the mean vorticity jump is non-zero and is directly related to the phase shift ϕ, as can be seen from (5.4) if $\mu = 0$. Instead, however, (5.4) determines the mean-velocity jump in terms of the phase shift ϕ. The above result also demonstrates that, concerning the mean-vorticity jump, a direct match with the steady solution is not possible. Nevertheless a connection with the steady solutions may be effected in all other main respects. For μ small, with $Y^* = O(1)$, to leading order the steady Haberman equation holds. For large Y^* this gives $\zeta^* \sim Y^* + 2H^{\pm} + O(1/Y^*)$ therefore, with $H^+ \neq H^-$. Hence, writing

$$\zeta^* = Y^* + \tilde{\zeta}_m(Y^*) + O\left(\frac{1}{Y^*}\right), \tag{5.5}$$

we have that $\tilde{\zeta}_m$ satisfies

$$-\mu\tilde{\zeta}_{mY^*} = \tilde{\zeta}_{mY^*Y^*}, \tag{5.6}$$

after substitution into (2.8) and integration with respect to X^* from $X^* = 0$ to $X^* = 2\pi$. Here (5.6) yields

$$\tilde{\zeta}_m = h_0 + h_1\,\mathrm{e}^{-\mu Y^*}. \tag{5.7}$$

If $\mu > 0$ the boundary conditions give $h_0 = 2H^-$ and $2H^+ = h_0 + h_1$, and if $\mu < 0$ they give $h_0 = 2H^+$ and $2H^- = h_0 + h_1$. Hence an outer region wherein $Y^* = O(1/\mu)$ is necessary, for μ small, to adjust the vorticity jump back to zero. This shows that the critical-layer time dependence spreads out like $Y^* \sim \mu^{-1}$ *above* the critical layer if $\mu > 0$, but *below* it if $\mu < 0$. Formally, for boundary-layer flow, if μ becomes as small as $O(\epsilon^{\frac{2}{3}})$ then the critical layer width $\epsilon^{\frac{23}{3}} Y^* \to O(\epsilon^7)$, and the time dependence then enters the outer inviscid zone IZ. Also, (3.12*b*) and (3.10*a*, *b*) show that the slow variations now become $O(\epsilon^{-5}(\epsilon^2 X_2))$ and $O(\epsilon^{-4}(\epsilon^2 T_2))$ in space and time respectively. Similar comments apply to channel flow and liquid-layer flow, where if $\mu \to O(\epsilon^{\frac{4}{3}})$ the X_2 and T_2 variations increase by a factor $\epsilon^{\frac{4}{3}}$, and the time dependence again spreads out into the outer zone.

We turn next to the limiting cases of (2.8). A balance of the inertial and viscous terms suggests that viscous or time-dependent effects are significant when either $\gamma_c \gg 1$ and $Y^* = O(\gamma_c^{\frac{1}{3}})$, or $\mu \gg 1$ and $Y^* = O(\mu^{\frac{1}{2}})$ respectively. By contrast, if μ and γ_c are both small then nonlinearity is important. We consider the different limiting regimes in turn below, but only briefly for now, since it is felt that subsequent numerical solutions of (2.8) would probably provide firmer evidence of the solution characteristics emerging in such limits.

5.1. *The linear viscous time-dependent critical layer: $\gamma_c \gg 1, \mu \gg 1$ with $\mu = O(\gamma_c^{\frac{2}{3}})$*

Let $Y^* = \gamma_c^{\frac{1}{3}}\eta$ and $\mu = \gamma_c^{\frac{2}{3}}\bar{\mu}$, where η and $\bar{\mu}$ are $O(1)$. The boundary conditions (5.2*a*–*c*) and (2.8) suggest an expansion of the form

$$\left.\begin{aligned} U^* &= \tfrac{1}{2}\gamma_c^{\frac{2}{3}}\eta^2 + 2\gamma_c^{\frac{1}{3}} H^{\pm}\eta + \tfrac{1}{3}\cos X \ln\gamma_c + U_1 + \gamma_c^{-\frac{2}{3}} U_2 + \gamma_c^{-\frac{5}{3}} U_3 + \dots, \\ \zeta^* &= \gamma_c^{\frac{1}{3}}\eta + \gamma_c^{\frac{1}{3}} 2H^{\pm} + \gamma_c^{-\frac{1}{3}}\zeta_1 + \gamma_c^{-1}\zeta_2 + \dots . \end{aligned}\right\} \tag{5.8}$$

Substitution in (2.8) gives the sequence of equations

$$-\bar{\mu}\zeta_{1\eta}-\zeta_{1\eta\eta}+\eta\zeta_{1X^*} = -\sin X^*, \tag{5.9a}$$

$$-\bar{\mu}\zeta_{k,\eta}-\zeta_{k,\eta\eta}+\eta\zeta_{k,X^*} = -\sin X^*\zeta_{k-1,\eta} \quad (k = 2,3,\ldots). \tag{5.9b}$$

Here if $\bar{\mu}$ is zero the equations are identical with those discussed by Haberman and can be solved similarly. For non-zero $\bar{\mu}$ analogous results for integral representations of the solutions of (5.9) are stated in the Appendix.

Making use of the results in the Appendix, we find

$$\left.\begin{aligned} \zeta_1 &= -\operatorname{Im}\left\{e^{iX^*}\int_0^\infty e^{-i\eta t}\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\,dt\right\} \\ U_1 &= -\operatorname{Im}\left\{e^{iX^*}\int_0^\infty \frac{e^{-i\eta t}-1}{-it}\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\,dt\right\}+f_1(X^*), \end{aligned}\right\} \tag{5.10}$$

where $U_1 = f_1$ at $\eta = 0$. The asymptotic expansion of U_1 for large η now gives

$$\begin{aligned} U_1 = -\operatorname{Im}\Bigg\{e^{iX^*}\Bigg[&-i\log|\eta|+\tfrac{1}{2}\pi\,\operatorname{sgn}(\eta) \\ &+i\int_0^\infty \frac{\cos t-e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}}{t}\,dt+O\left(\frac{1}{\eta}\right)\Bigg]+f_1(X)\Bigg\} \quad \text{as } \eta\to\pm\infty, \end{aligned} \tag{5.11}$$

where $\operatorname{sgn}(\eta) = \pm 1$ for $\eta \gtrless 0$. In particular, (5.11) determines the leading-order velocity jump across the critical layer:

$$U_1^+-U_1^- = -\operatorname{Im}(e^{iX}\pi). \tag{5.12}$$

The solution for ζ_2 is given by

$$\begin{aligned} \zeta_2 = -\tfrac{1}{2}\operatorname{Im}&\left\{\int_0^\infty \frac{i\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}}{\bar{\mu}-it}\,e^{-i\eta t}\,dt\right\} \\ &-\tfrac{1}{4}\operatorname{Im}\left\{e^{2iX^*}\int_0^\infty e^{-i\eta t}\,e^{-(\frac{1}{6}t^3+\frac{1}{4}i\bar{\mu}t^2)}\int_0^t \tau\,e^{-(\frac{1}{6}\tau^3+\frac{1}{4}i\bar{\mu}\tau^2)}\,d\tau\,dt\right\}. \end{aligned} \tag{5.13}$$

For $\bar{\mu} \neq 0$ (5.13) shows that there is no vorticity jump, as required by (5.3), but there is a jump in the mean velocity.

Similarly ζ_3 can be determined, and after some tedious but straightforward algebra we find that

$$\begin{aligned} \zeta_3 = \operatorname{Im}\Bigg\{e^{iX^*}\Bigg[&\frac{1}{4}\int_0^\infty e^{-i\eta t}\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\,t\,dt \\ &+\tfrac{1}{4}i\bar{\mu}\int_0^\infty [\log(i\bar{\mu})-\log(t+i\bar{\mu})]\,e^{-i\eta t}\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\,dt \\ &+\frac{1}{8}\int_0^\infty e^{-i\eta t}\,e^{-(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\int_0^t \tau\,e^{+\frac{1}{6}\tau^3+\frac{1}{4}i\bar{\mu}\tau^2}\int_0^\tau T\,e^{-(\frac{1}{6}T^3+\frac{1}{4}i\bar{\mu}T^2)}\,dT\,d\tau\,dt\Bigg]\Bigg\} \\ &-\tfrac{1}{24}\operatorname{Im}\left\{e^{3iX^*}\int_0^\infty e^{-i\eta t}\,e^{-(\frac{1}{9}t^3+\frac{1}{6}i\bar{\mu}t^2)}\int_0^t \tau\,e^{-(\frac{1}{18}\tau^3+\frac{1}{12}i\bar{\mu}\tau^2)}\int_0^\tau T\,e^{-(\frac{1}{6}T^3+\frac{1}{4}i\bar{\mu}T^2)}\,dT\,d\tau\,dt\right\} \\ &-\tfrac{1}{4}\operatorname{Im}\Bigg\{e^{-iX^*}\Bigg[\int_{-\infty}^0 e^{-i\eta t}\,e^{\frac{1}{6}t^3+\frac{1}{2}i\bar{\mu}t^2}\,dt\int_0^\infty \frac{t}{t+i\bar{\mu}}\,e^{-2(\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2)}\,dt \\ &-\int_0^\infty e^{-i\eta t}\,e^{\frac{1}{3}t^3+\frac{1}{2}i\bar{\mu}t^2}\int_\infty^t \frac{\tau}{\tau+i\bar{\mu}}\,e^{-2(\frac{1}{3}\tau^3+\frac{1}{2}i\bar{\mu}\tau^2)}\,d\tau\,dt\Bigg]\Bigg\}. \end{aligned} \tag{5.14}$$

Then U_3 can be obtained by integration, and an asymptotic expansion for large η yields a correction for the phase shift ϕ. Hence, collecting the above results together, we find that

$$\phi = -\pi + \tfrac{1}{2}\gamma_c^{-\frac{4}{3}}\pi \int_0^\infty \frac{t^2\cos(\bar{\mu}t^2) - \bar{\mu}t\sin(\bar{\mu}t^2)}{t^2+\bar{\mu}^2}\, e^{-\frac{2}{3}t^3}\, dt + \dots \tag{5.15}$$

As $\bar{\mu}\to 0$, with η fixed, the above expressions reduce to those given by Haberman (1976), although, as discussed above, for any non-zero $\bar{\mu}$ (2.8) gives no vorticity jump whereas the Haberman equation entails a discontinuity in the vorticity across the critical layer. This apparent inconsistency, however, can be identified with a closer examination of the mean-flow term in (5.13), which when rewritten is

$$\zeta_2^{\mathrm{m}} = -\frac{1}{2}\int_0^\infty \frac{[\bar{\mu}\cos(t\eta+\frac{1}{2}\bar{\mu}t^2) + t\sin(t\eta+\frac{1}{2}\bar{\mu}t^2)]}{\bar{\mu}^2+t^2}\, e^{-\frac{1}{3}t^3}\, dt. \tag{5.16}$$

For small $\bar{\mu}$

$$\zeta_2^{\mathrm{m}} \sim -\frac{1}{2}\left[\int_0^\infty \frac{\cos(\bar{\mu}t\eta)}{1+t^2}\, dt - \frac{1}{\bar{\mu}}\frac{\partial}{\partial\eta}\int_0^\infty \frac{\cos(\bar{\mu}t\eta)}{1+t^2}\, dt\right] + \dots$$

Now

$$\int_0^\infty \frac{\cos\sigma t}{1+t^2}\, dt = \tfrac{1}{2}\pi\, e^{-|\sigma|};$$

hence

$$\zeta_2^{\mathrm{m}} \sim -\tfrac{1}{4}\pi\,[e^{-|\bar{\mu}\eta|} - \mathrm{sgn}(\eta)\, e^{-|\bar{\mu}\eta|}]. \tag{5.17}$$

For $|\eta| \gg 1/\bar{\mu}$ we recover the zero vorticity jump, and for η large, but $|\eta\bar{\mu}| \ll 1$, $\zeta_2^{\mathrm{m}} \sim -\frac{1}{4}\pi[1-\mathrm{sgn}(\eta)]$, giving a jump of $\frac{1}{2}\pi\gamma_c^{-1}$ in the vorticity, as in the steady case. Thus for $\bar{\mu}$ small an outer region, where η is $O(1/\bar{\mu})$, is necessary to adjust the vorticity jump to zero, and this agrees with the earlier comments concerning the emergence of the steady solution as $\mu\to 0$.

If in the above we retain the T^* dependence, i.e. we consider the linear solutions of the general unsteady equation (2.9), we obtain

$$\zeta_1 = -\mathrm{Im}\left\{e^{iX^*}\int_0^{\tilde{T}} e^{-i\eta u}\, e^{-(\frac{1}{3}u^3+\frac{1}{2}i\bar{\mu}u^2)}\, du\right\},$$

where $\tilde{T} = \gamma_c^{\frac{1}{3}} T^*$. In particular, the velocity jump is given by

$$U_1^+ - U_1^- = \lim_{\eta\to\infty} -\mathrm{Im}\left\{2\, e^{iX^*}\int_0^{\tilde{T}\eta} \frac{\sin u}{u}\exp\left[-\left(\frac{u^3}{3\eta^3} + i\frac{u^2\bar{\mu}}{2\eta^2}\right)\right] du\right\}. \tag{5.12$'$}$$

As $|\eta\tilde{T}|\to\infty$ this reduces to the 'steady' limit discussed above. The results (5.12) and (5.12′), for our moving viscous critical layer, are the same as for a linear unsteady critical layer, as in Dickinson (1970), Stewartson (1981) and Hickernell (1984). The mean-flow term in ζ_2, again with the T^* dependence retained, is given by

$$\zeta_2^{\mathrm{m}} = \mathrm{Im}\left\{\frac{1}{2}\int_0^{\tilde{T}} e^{-i\eta u}\,\frac{e^{-(\frac{1}{3}u^3+\frac{1}{2}i\bar{\mu}u^2)}}{u+i\bar{\mu}}\,[1-e^{-(T-u)(u^2+i\bar{\mu}u)}]\, du\right\}.$$

For $\bar{\mu} = 0$ this is the same as equation (3.29) of Stewartson (1981) (after allowing for misprints), and for $|\eta\tilde{T}| \gg 1$, fixed $\bar{\mu}$, this again retrieves the steady result. These latter results indicate that (2.9) together with the amplitude equations (3.15) and (3.29) is only valid provided in effect $|Y^*T^*| \gg 1$ in the present contexts, a point made earlier in §2. The above expressions for the velocity jump and the mean flow in fact demonstrate that generally for $|Y^*T^*| = O(1)$ the unsteady properties, with $\partial/\partial T^* \neq 0$, need to be reconsidered both inside and outside the critical layer, which represents a rather more intricate general case.

5.2. *The linear inviscid time-dependent critical layer:* $\mu \gg 1, \mu \gg \gamma_c^{\frac{2}{3}}$

Let $Y^* = \mu^{\frac{1}{3}}\tilde{\eta}$, and $\gamma_c = \bar{\gamma}_c \mu^{\frac{2}{3}}$ but $\bar{\gamma}_c \ll 1$. Again (2.8) with (5.2*a–c*) indicates expansions of the form

$$\left.\begin{aligned} U^* &\sim \mu\hat{U}_0 + \mu^{\frac{1}{3}}\hat{U}_1 + \mu^{-\frac{1}{3}}\hat{U}_2 + \tfrac{1}{2}\mu^{-\frac{1}{3}} \ln \mu \cos X^* + \ldots, \\ \zeta^* &\sim \mu^{\frac{1}{3}}\zeta_0 + \zeta_1 + \mu^{-\frac{1}{3}}\zeta_2 + \ldots. \end{aligned}\right\} \tag{5.18}$$

Substitution into (2.8) yields the equations

$$\left.\begin{aligned} 1 - \zeta_{0\tilde{\eta}} + \tilde{\eta}\zeta_{0X^*} - \bar{\gamma}_c\, \zeta_{0\tilde{\eta}\tilde{\eta}} &= 0, \\ -\zeta_{1\tilde{\eta}} + \tilde{\eta}\zeta_{1X^*} - \bar{\gamma}_c\, \zeta_{1\tilde{\eta}\tilde{\eta}} &= 0, \\ -\zeta_{2\tilde{\eta}} + \tilde{\eta}\zeta_{2X^*} - \bar{\gamma}_c\, \zeta_{2\tilde{\eta}\tilde{\eta}} &= -\sin X^* \zeta_{0\tilde{\eta}}. \end{aligned}\right\} \tag{5.19}$$

If we write $\zeta_0 = \tilde{\eta} + \zeta_{00}$, then ζ_{00} satisfies $-\zeta_{00\tilde{\eta}} + \tilde{\eta}\zeta_{00X^*} - \bar{\gamma}_c\, \zeta_{00\tilde{\eta}\tilde{\eta}} = 0$ and $\zeta_{00} \to$ const as $\tilde{\eta} \to \pm\infty$. For any non-zero $\bar{\gamma}_c$ all non-trivial periodic solutions for ζ_{00} become unbounded as $|\tilde{\eta}| \to \infty$, and thus $\zeta_{00} = h_{00}$, where h_{00} is some constant. Similarly, since $\zeta_1 \to h_{10}$ (h_{10} constant) as $\tilde{\eta} \to \pm\infty$, $\zeta_1 = h_{10}$. Then the solution for ζ_2 can be obtained as in §5.1, and is given by

$$\left.\begin{aligned} \zeta_2 &= -\operatorname{Im}\left\{ e^{iX^*} \int_0^\infty e^{-i\tilde{\eta}t}\, e^{-(\frac{1}{3}\bar{\gamma}_c t^3 + \frac{1}{2}it^2)}\, dt \right\}, \\ \hat{U}_2 &= -\operatorname{Im}\left\{ e^{iX^*} \int_0^\infty \frac{e^{-i\tilde{\eta}t} - 1}{-it}\, e^{-(\frac{1}{3}\bar{\gamma}_c t^3 + \frac{1}{2}it^2)}\, dt \right) + \hat{g}_2(X^*), \end{aligned}\right\} \tag{5.20}$$

where $\hat{g}_2$ is some function of X^* which can be determined from the boundary conditions. For $\bar{\gamma}_c \neq 0$, then, we obtain the same velocity jump as before:

$$\hat{U}_2^+ - \hat{U}_2^- = -\pi \sin X^*, \tag{5.21}$$

together with a zero vorticity jump. The limit $\bar{\gamma}_c \to 0$, $\tilde{\eta} \to \pm\infty$ is interesting and shows the existence of an outer region *below* the critical layer necessary to reduce the vorticity jump to zero. For if we change variables in (5.20) we have

$$\zeta_2 = -\operatorname{Im}\left\{ \gamma_c^{-\frac{1}{3}}\, e^{iX^*} \int_0^\infty e^{-\frac{1}{3}t^3}\, e^{-i(\tilde{\eta}t\gamma_c^{-\frac{1}{3}} + \frac{1}{2}t^2\gamma_c^{-\frac{2}{3}})}\, dt \right\}. \tag{5.22}$$

As $\tilde{\eta} \to +\infty$ there are no stationary points in the integrand of (5.22), and the major contribution occurs at the endpoint $t = 0$, giving $\zeta_2 = O(1/\tilde{\eta})$. In contrast, as $\tilde{\eta} \to -\infty$ the dominant contribution to ζ_2 comes from the stationary point at $t = -\tilde{\eta}\gamma_c^{\frac{1}{3}}$, and this gives

$$\zeta_2 = -\operatorname{Im}\left\{ e^{i(X^* + \frac{1}{2}\tilde{\eta}^2)}\, e^{\frac{1}{3}\tilde{\eta}^3\bar{\gamma}_c}\, 2^{\frac{1}{2}}\pi\, e^{-\frac{1}{4}i\pi} \right\} + O\left(\frac{1}{\tilde{\eta}}\right). \tag{5.23}$$

If $\eta^* = X^* + \frac{1}{2}\tilde{\eta}^2$ is fixed, $|\tilde{\eta}| \gg 1$ and $|\tilde{\eta}| \ll \bar{\gamma}_c^{-\frac{1}{3}}$, then (5.23) shows that there is a jump in the vorticity with

$$\zeta_2^+ - \zeta_2^- = \operatorname{Im}\,(e^{i\eta^*}\, 2^{\frac{1}{2}}\pi\, e^{-\frac{1}{4}i\pi}).$$

If, however, $|\tilde{\eta}| \gg \bar{\gamma}_c^{-\frac{1}{3}}$ then there is no difference in the mean vorticity above and below the critical layer. Hence for η^* fixed there is an outer region below the critical layer, with $Y^* = O(\mu\gamma_c^{-\frac{1}{3}})$ in terms of the original variables, in which vorticity diffusion is important.

5.3. *The time-dependent nonlinear critical layer:* $\mu = O(1)$, $\gamma_c \ll 1$

To leading order, from (2.8) ζ^* now satisfies

$$\mu(1-\zeta^*_{Y^*})+Y^*\zeta^*_{X^*}+\sin X^*\zeta^*_{Y^*}=0, \tag{5.24}$$

for some μ. Hence, formally at least, and in brief,

$$\zeta^* = \int_{\chi\,\text{fixed}}^{X^*} \frac{-\mu\,\mathrm{d}X'}{2^{\frac{1}{2}}[\chi-\mu X'-\cos X']^{\frac{1}{2}}}+K(\chi), \tag{5.25}$$

where $\chi = \frac{1}{2}Y^{*2}+\mu X^*+\cos X^*$ is the basic stream function, and $K(\chi)$ is an unknown function of χ. The inviscid equation (5.24) is the same in essence as one of Cowley's (1981), who derives and discusses various results.

The basic streamlines given by χ = constant are sketched in figures 3(*a*,*b*). For $\mu \geqslant 1$ all the streamlines are *open*, whereas for $\mu < 1$ there is a train of distinct *closed* eddies with open streamlines in-between. With χ fixed the integral in (5.25) is assumed to be periodic for all μ. Certainly for μ large the results of §5.2 in the limit $\bar{\gamma}_c \rightarrow 0$ are recovered; and for μ small the integral is again periodic. It is suggested tentatively that (for some μ) a viscous correction to (5.25), combined with a 2π-periodicity requirement, may then determine the unknown vorticity function $K(\chi)$, and this gives for open streamlines

$$\int_0^{2\pi} \frac{\partial}{\partial\chi}\left[(\chi-\mu x-\cos x)^{\frac{1}{2}}\frac{\partial\zeta^*}{\partial\chi}(x,\chi)\right]\mathrm{d}x = 0, \tag{5.26}$$

provided the integral exists (otherwise the integration range must be altered), where $\zeta^*(x,\chi)$ is defined by (5.25). For closed streamlines a singularity-free condition must hold, as in previous works. The working leading to the tentative criterion (5.26) for determining $K(\chi)$ has some connection with Cowley's (1981) work, incidentally, as a referee has pointed out, although the latter study largely omits viscous effects. In fact for $\mu = 0$ (5.26) is consistent with the results of Haberman (1972) and Smith & Bodonyi (1982*a*) for the limit $\gamma_c \rightarrow 0$. An explicit determination of the function $K(\chi)$ and more significantly of the velocity jump resulting from (5.25) is left unresolved here, however; it is expected that future numerical results for the problem (2.8) will indicate more clearly what the trends are in this important regime. This regime is particularly interesting because it corresponds to increased disturbance sizes. Likewise the regime of §5.2 is of physical importance since it is associated with faster timescales. Both regimes therefore merit further analytical and numerical study to gain more insight into the evolution and movement of time-dependent nonlinear critical layers.

6. A brief summary

We would emphasize the following points concerning the present study.

(*a*) In general the global space–time evolution of the disturbances considered is controlled by the nonlinear equations presented at the ends of §3.1 and 3.2. These form a perhaps novel set coupling the slow-scale dependence (X_2, T_2) (of the amplitude A and wavespeed c_0) non-linearly with the faster-scale dependence (X_1, T_1) (of the amplitude) and with the critical-layer movement (via the unknown nonlinear function $\mu(A)$ and the resultant phase shift $\phi(A)$). It is necessary therefore that in general solutions of the time-dependent nonlinear critical-layer problem (2.8) be available, a task requiring numerical work. The limiting solutions in §5 then provide some preliminary comparisons for the numerical work which should be of interest.

(*b*) The special case of fixed-frequency disturbances discussed in §4 is encouraging

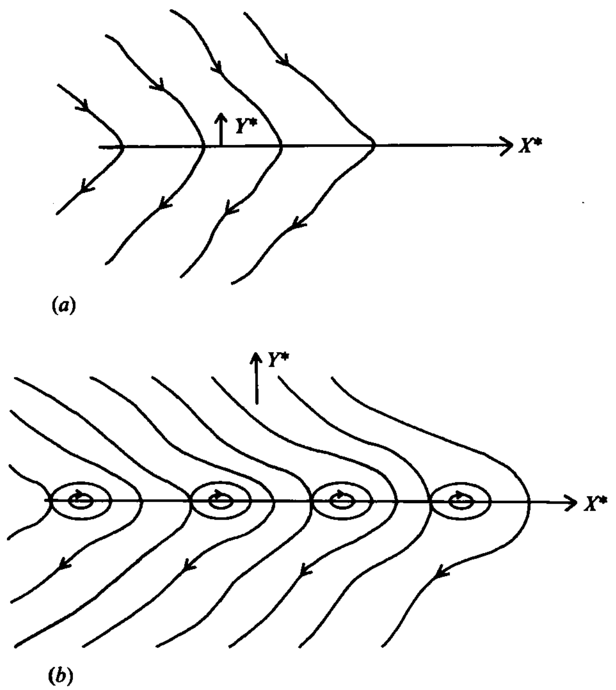

FIGURE 3. Sketch of the basic streamlines of $\chi = \frac{1}{2}Y^{*2} + \mu X^* + \cos X^* = \text{const}$; see §5.3: (a) $\mu \geqslant 1$; (b) $\mu < 1$.

for further research. It shows that the previous quasi-neutral estimates (e.g. Smith & Bodonyi 1982*a*) produce, in effect, subcritical threshold amplitudes for given frequencies. An amplitude initially below the threshold value subsequently decays, whereas nonlinear growth occurs for initial amplitudes above the threshold value. Moreover, the growth is unbounded in the present terms and can be analysed then, until a new structure subsequently comes into play. Whether similar fast growth can occur more generally for varying frequency, wavespeed and wavenumber, or not, remains to be seen from the general equations referred to in (*a*) above, and this in turn supports the value of a numerical study of (2.8). If fast growth is induced, then the limits studied in §5 come back into play, and these should point ultimately to the next stage in the evolution of the growing disturbance, possibly bringing the full Euler equations into the final reckoning.

(*c*) The equation governing the effective local wavespeed of the wave packet is the inviscid Burger equation (3.15*c*) or (3.29*c*). Hence shocklike behaviour can take place, which may be significant. Moreover, since the wavespeed is real to leading order, another generalization of the global evolution equations can be made which adds a term $\propto \partial^2 A/\partial x^2$, to (4.2) for instance. This would increase the relevance of the theory to initial-value problems.

(*d*) We notice that the time-dependent governing equations, derived here from formal substitution into the Navier–Stokes equations, are contained also within the interacting boundary-layer framework of Smith *et al.* (1984), for finite *Re*. Thus the nonlinear instability features indicated analytically in this paper should be representable at finite *Re* within that framework.

(*e*) Three-dimensional disturbances seem especially worthy of investigation along similar lines, to increase the possible relevance to observed instability processes in boundary layers and channel flows for instance. Larger disturbances are also a major concern. Again, the structures and properties discussed here may be adapted to the study of non-parallel basic flows, which force added movement of the critical layer across the flow, and to stratified fluid flows.

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Appendix

Consider the equation

$$\eta\zeta_X - \zeta_{\eta\eta} - \mu\zeta_\eta = \sum_{\omega=-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(\omega, t)\, e^{-it\eta}\, dt, \tag{A 1}$$

where the right-hand side is some known periodic function of x. For periodic solutions the related homogeneous equation

$$i\omega\eta\zeta - \zeta_{\eta\eta} - \mu\zeta_\eta = 0 \tag{A 2}$$

has only solutions that become unbounded as $\eta \rightarrow \pm\infty$. Hence the bounded solution to the forced equation (A 1) is given uniquely by

$$\zeta = \sum_{\omega=-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(\omega, t)\, S(\eta, t, \omega)\, dt,$$

where

$$S(\eta, t, \omega) = \frac{1}{\omega} e^{t^3/3\omega + i\mu t^2/2\omega} \int_t^{\infty} e^{-i\eta t}\, e^{-(t^3/3\omega + it^2\mu/2\omega)}\, dt \quad (\omega < 0),$$

$$= -\frac{1}{\omega} e^{t^3/3\omega + i\mu t^2/2\omega} \int_{-\infty}^{t} e^{-i\eta t}\, e^{-(t^3/3\omega + it^2\mu/2\omega)}\, dt \quad (\omega < 0). \tag{A 3}$$

The latter results are derived as in Haberman (1976).

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